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Plane mechanics and kinematics of compressible ideal granular materials

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1. Introduction

The question of the correct formulation of constitutive equations to describe the mechanical and kinematic behaviour of granular materials such as soils and powders remains one of discussion and controversy. The purpose of this paper is to make some contributions to this discussion with particular reference to compressible materials. To this end we put forward some suggestions for consideration, while recognising the existence of alternative approaches to the problems discussed, and of differences of opinion which remain to be resolved.

In a previous paper Spencer (1) made proposals for the equations which govern the mechanical and kinematic behaviour in plane strain of an ideal incompressible granular material. The author subsequently learned that essentially the same equations had been formulated earlier by Mandl (2), although Mandl (3) has since raised objections to the theory which leads to these equations. The theory is based on material isotropy, incompressibility, the Coulomb yield condition (although it can readily be extended to any other yield function which is a function of the two invariants of the stress tensor in two dimensions) and, more controversially, an assumption that deformation takes place by simultaneous shears on the characteristic curves of the stress equations. The predictions of the resulting theory do not appear to conflict violently with observations of the behaviour of some granular materials which are to a reasonable approximation incompressible. However, there are alternative theories available, some of which were described in (1). Space does not permit discussion of more recent work. However, we mention papers by Dais (4), Mandl and Fernandez Luque (5) [see also the discussion by Spencer (6)], de Josselin de Jong (7), and Goodman and Cowin (8, 9), and take this opportunity to record our opinion that the best prospect for future progress lies in investigating the predictions of the various theories which have been proposed with a view to devising critical experiments to test them against one another.

Although some granular materials show only small volume changes in the range of stress states they encounter in practice, for others compressibility effects are certainly important. In this paper we attempt to extend to compressible materials some of the theory developed in (1). It is emphasized that these suggestions are put forward in a tentative manner, as a basis for discussion. They certainly do not amount to a complete theory. In the first place they deal only with plane deformations, and it has yet to be shown that the plane deformation theory can be embedded within a full three-dimensional theory. Second, we do not consider the relative motions and force interactions between the constituents which make up the material, and we judge these to be of importance in many situations. Third, the theory retains many features of the theory described in (1) which remain open to verification or otherwise. In spite of these uncertainties, we hope that this paper will make some contribution to the eventual formulation of a satisfactory and accepted theory to describe the mechanical behaviour of granular materials.

2. Densities and Specific Volumes

We envisage the granular material as comprising a mixture of two materials, although obviously more complex systems are possible. It is supposed that one of the phases is a solid phase, and the other is a gas. This approximates the situation in a dry powder or dry sand.

The density of the mixture (mass of mixture per unit volume of mixture) is denoted by \( q \). Its specific volume (volume of mixture per unit mass of mixture) is then \( q^{-1} \). The mass concentration of the solid phase (mass of solid per unit mass of mixture) is \( c \), and that of the gas phase is then \( 1 - c \). The density of solid (mass of solid per unit volume of mixture) is \( c q \) and the density of gas is \( (1 - c) q \). The volume concentration of the solid phase (volume of solid per unit volume of mixture) is denoted \( v \), and that of the gas phase is then \( 1 - v \). The specific volume of the solid phase (volume of solid per unit mass of mixture) is \( v q^{-1} \), and that of the gas phase is \( (1 - v) q^{-1} \). In practice we expect the density of the gas to be much less than that of the solid, so that

\[
(1 - c) \ll 1, \quad c \simeq 1 \quad [2.1]
\]

but no similar relations need hold for \( v \). In order to describe the density and composition of an element of the mixture at any time, it is necessary to specify all three of \( q, c \) and \( v \), or three equivalent quantities; any one of \( q, c \) and \( v \) may in principle be varied independently of the other two.
Although we have identified the two constituents of the material, we are constructing a continuum theory and, as for example in the mathematical theory of mixtures, we do not distinguish the places occupied by the solid and gas. Each constituent is regarded as distributed continuously throughout the volume occupied by the material, and each point of the body as simultaneously occupied by both solid and gas particles. A more rigorous approach to this formulation is employed by Goodman and Cowin (9).

It is to be expected that normally the solid phase will be much less compressible than the gas phase. If we idealize this property by assuming that the mass of solid per unit volume of solid is a constant \( c_0 \) (which may be interpreted as the density of the granules in a real granular material) then

\[
\gamma = v \gamma. \tag{2.2}
\]

Thus in this case one of \( \gamma, c, v \) may be expressed in terms of the other two.

### 3. Freely-diffusing and non-diffusing materials

In order to investigate the relations between the variables \( \gamma, c \) and \( v \) and the forces acting in a body it would appear to be necessary to introduce separate velocity fields for each constituent, partial stress fields corresponding to each constituent, and diffusive forces arising from the relative motions of the constituents. There are, however, two cases which appear to be of interest in which these complications can to some extent be avoided.

**Freely-diffusing Materials**

We say the material is freely-diffusing if the gas phase can move freely relative to the solid phase, or, more exactly, if deformations of the material take place over times long compared to a characteristic time for gas to diffuse through the material. In such a situation it may be assumed that the pore pressure is constant in the material. If the pore pressure is also assumed to be a function of the pore density (defined as mass of gas per unit volume of gas) which has the value \( (1 - c) \gamma / (1 - v) \), then in a freely diffusing mixture

\[
(1 - c) \gamma / (1 - v) = \gamma_0 \tag{3.1}
\]

where the constant \( \gamma_0 \) may be taken to be the ambient density outside the material.

**Non-diffusing Materials**

The other extreme case of a non-diffusing material arises when there is no relative motion between the gas and solid constituents, or when the time over which deformation of the material takes place is short compared to a characteristic time for gas to diffuse through the material. In this case the mass concentrations at a particle of the mixture are constant, and so

\[
c = c_0 \tag{3.2}
\]

where \( c_0 \) is constant at a particle. Also in this case the velocity field is unambiguously defined for the mixture, and there is no need to distinguish velocity fields for the individual constituents.

### 4. Flow condition

We consider two-dimensional states of stress and deformation which we refer to rectangular cartesian coordinates \((x, y)\). Stress components in the \((x, y)\) plane are denoted by \( t_{xx}, t_{yy}, t_{xy} \), and tensile stress components are taken to be positive. The two principal stress components whose principal directions lie in the \((x, y)\) plane are denoted by \( t_1, t_2 \) and without loss of generality we take \( t_1 \geq t_2 \). Define

\[
p = -\frac{1}{2}(t_1 + t_2), \quad q = \frac{1}{2}(t_1 - t_2). \tag{4.1}
\]

Then

\[
t_{xx} = -p + q \cos 2\psi \\
t_{yy} = -p - q \cos 2\psi \\
t_{xy} = q \sin 2\psi \tag{4.2}
\]

\[
p = -\frac{1}{2}(t_{xx} + t_{yy}) \\
q = \left(\frac{1}{4}(t_{xx} - t_{yy})^2 + t_{xy}^2 \right)^{\frac{1}{2}} \tag{4.3}
\]

where \( \psi \) is the angle of inclination, measured anticlockwise, of the \( t_1 \) principal stress direction to the \( x \)-axis. These stress components are all total stress components for the mixture. Partial stress components for the solid and gas will not be explicitly introduced.

We follow the usual procedure of metal and soil plasticity and postulate the existence of a function of the stress components which is equal to zero when the material is deforming and negative or zero otherwise. For isotropic material this function must be expressible as a function of the stress invariants \( p \) and \( q \). Since we are considering a compressible material we also admit dependence on the density \( \gamma \). It is assumed that \( \gamma_0 \) and either [3.1] or [3.2] hold, so that \( c \) and \( v \) can