On Exponents in Arithmetical Semigroups

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Abstract

Let \( n = \prod_{i=1}^{r} p_i^{x_i} \) be the factoring of a natural number \( n \) into primes. In 1969 Niven found the average order of the functions \( h(n) = \min \{x_1, \ldots, x_r\} \) and \( H(n) = \max \{x_1, \ldots, x_r\} \). In this note a generalization of Niven's results is discussed, namely the average order of the corresponding functions in certain arithmetical semigroups.

A free Abelian semigroup \( G = \{n, m, \ldots\} \) with identity and a countable system of generators \( E = \{p_1, p_2, p_3, \ldots\} \) will be called arithmetical if to every element \( n \in G \) a real-valued norm \( |n| \) is assigned such that

(a) \( |n \cdot m| = |n| \cdot |m| \) for all \( n, m \in G \),
(b) \( |p_i| > 1 \) for all \( i = 1, 2, \ldots \),
(c) \( \lim_{i \to \infty} |p_i| = \infty \).

We borrowed this definition from Wegmann [2], but Wegmann used the name \( F \)-Halbgruppe.

If \( A \) is a subset of \( G \), then we put

\[ A(x) = \sum_{n \in A, |n| < x} 1, \]

in particular

\[ G(x) = \sum_{n \in G, |n| < x} 1. \]

The counting function \( G(x) \) will be called \( \delta \)-regular [2] if

\[ G(x) = x^\delta L(x), \]

\(^1\) It will be clear from the context whether \( |\cdot| \) means the absolute value or this norm.
where \( L(x) \) is defined for all \( x > 0 \) and
\[
\lim_{x \to \infty} \frac{L(ax)}{L(x)} = 1 \text{ for all } a > 0.
\]

The set of natural numbers and the set of non-zero integral ideals of a finite extension of degree \( n \) of the field of rational numbers are the most known examples of arithmetical semigroups with \( \delta \)-regular \( G(x) \), more precisely with \( G(x) = x + O(1) \) and \( G(x) = cx + O(x^{(n-1)/(n+1)}) \), resp.

In an arithmetical semigroup \( G \) we can define the zeta-function in a usual way
\[
\zeta_G(s) = \sum_{n \in \mathbb{G}} \frac{1}{n^s}.
\]

According to Satz 1.7 of [2] \( \zeta_G(s) \) is convergent for \( s > \delta \) and divergent for every \( s < \delta \), provided \( G(x) \) is \( \delta \)-regular. This follows among others from the fact (see [2]) that if \( G(x) \) is \( \delta \)-regular, then
\[
\lim_{x \to \infty} \frac{G(x)}{x^{\delta+\varepsilon}} = 0 \quad \text{and} \quad \lim_{x \to \infty} \frac{G(x)}{x^{\delta-\varepsilon}} = \infty \tag{1}
\]
for every \( \varepsilon > 0 \).

Since \( G \) is a free semigroup, every element \( n \) of \( G \) can be factorized uniquely into a product of generators, say,
\[
n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}.
\]

Thus we can define the Möbius function
\[
\mu_G(n) = \mu(n) = \begin{cases} 1, & \text{if } n \text{ is the identity of } G, \\ (-1)^r, & \text{if } \alpha_1 = \ldots = \alpha_r = 1, \\ 0, & \text{otherwise} \end{cases}
\]
and also the functions
\[
h_G(n) = \min \{\alpha_1, \ldots, \alpha_r\},
\]
\[
H_G(n) = \max \{\alpha_1, \ldots, \alpha_r\}.
\]

In Theorem 1 we shall need the following function [2]
\[
r(x,a) = \frac{a^\delta G(x/a)}{G(x)} - 1.
\]