BOUNDARY CONDITIONS IN THE HYDROMAGNETIC DYNAMO PROBLEM

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1. INTRODUCTION

The theory of the hydromagnetic dynamo deals with the questions of regeneration of the magnetic fields of the Earth, Sun and other Universal bodies. The problem rests in solving the equation of magnetic induction

\[ \frac{\partial B}{\partial t} = R_m \text{rot} (u \times B) + \Delta B \]

and the equation of motion of a fluid (the Navier-Stokes equation) in a conducting moving medium of spherical shape. The vector of magnetic induction \( B \) and the velocity of the fluid are transformed in a dimensionless system of units in (1), \( R_m \) is the dimensionless Reynolds magnetic number. The dimensionless system of units is usually introduced by taking the radius of the sphere \( R \) to be equal to unity.

At boundary \( V \), \( B \) must be continuous with the external field which has a potential character (rot \( B = 0 \)) and decreases to zero with increasing distance. The boundary condition is not expressed in a form usual for partial differential equations and this fact influenced the selection of the methods of solution. In analysing them, we shall restrict ourselves to the kinematic model, i.e. to solving (1) for a given velocity, because the condition \( u = 0 \) is usually satisfied at the boundary and, therefore, from the point of view of formulating the boundary conditions, the transition from the kinematic to the hydromagnetic model represents no difficulties.

With a view to the shape of region \( V \), the problem has usually been treated in spherical coordinates \( (r, \theta, \phi) \). The hitherto most widely used method, proposed in [1] and developed in [2], is based on resolving vectorial fields into their toroidal and poloidal parts,

\[ B = T + S = \text{rot} \left( \frac{T(r)}{r} \right) i_r + \text{rot}^2 \left( \frac{S(r)}{r} \right) i_r \]

and in expanding these components into a series of spherical harmonics

\[
S = \sum_{n=1}^{\infty} \sum_{m=1}^{n} \left[ S_n^{nc}(r, t) \cos (m\phi) + S_n^{ms}(r, t) \sin (m\phi) \right] P_n^m(\cos \theta)
\]

Similarly for \( T \). The boundary condition at the boundary \( r = 1 \) will then read \( T_n^m = (\partial S_n^m/\partial r) + 1 + nS_n^m = 0, r = 1 \). According to [3], the simple formulation of the boundary condition was the
main reason for this method of solution. Other reasons are sure to have been the similarity with the spherical harmonic analysis of the geomagnetic field and the general trend in mathematics of the fifties "to solve differential equations by expansion into orthogonal series". These reasons were sufficiently viable to render this procedure the only method of solving models of the kinematic dynamos until the middle of the seventies. The first method without expansion into a series of spherical functions was only published by Jepps [4] in 1975 and another a year later by Ivanova [5]. Both methods were developed for the model of the turbulent dynamo, however, they can be used everywhere, where just the symmetric part of the field \( B \) enters into the regeneration equations, i.e. also for models of the nearly symmetric dynamo. Since quite new numerical procedures are involved, the papers lack such numerical results which would enable the convergence of both methods to be compared. Jepps and Ivanova treat the parabolic problem, so that they only verify the approximate agreement with earlier calculations without being able to evaluate the differences precisely. Moreover, the results are presented in the form of graphs and the calculations were made for a single net density.

The author of the present paper has, therefore, made a detailed comparison, using a problem whose solution is known in analytical form. In this way the errors of the eigenvalues and eigenvectors could be determined exactly. Since the calculations showed that Ivanova's process converges much more slowly, the author, drawing on [6], attempted to improve this convergence. The description of the new method and the comparison of all three procedures are the subject of this paper.

2. MODIFICATION OF THE REGENERATION EQUATIONS; TEST PROBLEM

If \( B \) is axially symmetrical (it only depends on the spatial co-ordinates \( r \) and \( \vartheta \)), the toroidal and poloidal components can be expressed as follows:

\[
T = \frac{T}{r} i_\varphi, \quad S = \text{rot} \frac{S}{r} i_\varphi.
\]

The sphere \( V \) reduces to the semi-circular region \( H = \{(r, \vartheta), 0 \leq r \leq 1, 0 \leq \vartheta \leq \pi\} \) and Eq. (1) reduces to the system

\[
\frac{\partial T}{\partial t} - LT = F_1 T + G_1 S, \quad \frac{\partial S}{\partial t} - LS = F_2 T + G_2 S,
\]

where

\[
L = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2 \sin \vartheta} \frac{\partial}{\partial \vartheta} \left( \sin \vartheta \frac{\partial}{\partial \vartheta} \right) - \frac{1}{r^2 \sin^2 \vartheta},
\]

\( F_i \) and \( G_i \) being linear partial differential operators of the 1st order, the actual form of which depends on the model being used. The toroidal field does not penetrate outside \( V \), whereas the poloidal field satisfies the equation

\[
LS = 0.
\]

Due to the continuity of \( B \) at boundary \( V \), we arrive at the conditions

\[
T = [S] = [\partial S/\partial r] = 0, \quad r = 1,
\]

where \([\ ]\) indicates a leap across the boundary. Along the axis of symmetry it must hold that