Polynomial Behaviour of Scattering Amplitudes at Fixed Momentum Transfer in Theories with Local Observables

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Abstract. It is shown that, in theories of exactly localized observables, of the type proposed by Araki and Haag, the reaction amplitude for two particles giving two particles is polynomially bounded in $s$ for fixed momentum transfer $t < 0$. The proof does not need observables localized in space-time regions of arbitrarily small volume, but uses relativistic invariance in an essential way. It is given for the case of spinless neutral particles, but is easily extendable to all cases of charge and spin. The proof can also be generalized to the case of particles described by regularized products

$$f_{\Phi}(x_1 \ldots, x_n) \phi_A(x - x_1) \ldots \phi_A(x - x_n) \, dx_1 \ldots dx_n$$

of Wightman or Jaffe fields.

Introduction

This paper studies two-particle reaction amplitudes in a theory of local observables of the type proposed by Araki and Haag [1--4]; it shows that, for fixed momentum transfer $t < 0$, such amplitudes are polynomially bounded functions of $s$ (square of total energy in centre-of-mass system). In ordinary field theory, the proof of this well-known result uses in an essential way the assumption that the vacuum expectation values of the fields behave polynomially at infinity. Although this assumption seems very reasonable, and is believed to be verified in renormalizable theories, it is satisfactory that the result can be derived from the independent hypotheses of the Araki-Haag theory. Such a theory could exist without fields in the ordinary sense, but it can also be considered as underlying any conventional field theory where local observations are possible (perhaps as a consequence of the self-adjointness of some smeared field operators). The framework of a theory of local observables can be briefly described as follows:

1. The physical state vectors are elements of a Hilbert space $\mathcal{H}$ in which operates a unitary, weakly continuous, representation of the Poincaré group $\mathcal{P}$, denoted by $(a, A) \to U(a, A)$, with $U(a, 1) = \exp i a \mu \mathbf{P}_\mu$. 

The momentum operators \( P^\mu \) are supposed to have their spectrum in \( \mathbb{R}^4 \), the closure of
\[
V^+ = \{ p \in \mathbb{R}^4, p^0 > |p| \} = - V^-.
\]

There is a vector \( \Omega \) (with \( \|\Omega\| = 1 \)), unique up to a phase factor, such that, for all \((a, A) \in \mathcal{P}^+_4\), \( U(a, A) \Omega = \Omega \) (vacuum); all state vectors orthogonal to \( \Omega \) have masses larger than a certain strictly positive minimum mass \( m_0 > 0 \).

1. To each open set \( \mathcal{O} \) in \( \mathbb{R}^4 \) (= Minkowski space = space-time) is associated a von Neumann algebra \( \mathcal{A}(\mathcal{O}) \), consisting of bounded operators acting in \( \mathcal{H} \), with the following properties
   a) if \( \mathcal{O}_1 \subset \mathcal{O}_2 \), then \( \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2) \);
   b) if \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) are spacelike separated (i.e., if \( x_1 \in \mathcal{O}_1 \) and \( x_2 \in \mathcal{O}_2 \) \( \Rightarrow (x_2 - x_1)^2 < 0 \)), then \( \mathcal{A}(\mathcal{O}_1) \) and \( \mathcal{A}(\mathcal{O}_2) \) commute;
   c) for every open \( \mathcal{O} \) and every \((a, A) \in \mathcal{P}^+_4\),
   \[
   U(a, A) \mathcal{A}(\mathcal{O}) U(a, A)^{-1} = \mathcal{A}(a + A \mathcal{O})
   \]
   d) \( \bigcup_{\mathcal{O} \text{ bounded}} \mathcal{A}(\mathcal{O}) \Omega \) is dense in \( \mathcal{H} \).

These algebras are called "algebras of local observables". An operator belonging to \( \mathcal{A}(\mathcal{O}) \) with \( \mathcal{O} \) bounded is called a local operator and is said to be localized in \( \mathcal{O} \).

A local ARAKI-HAAG field will be defined, in this paper, as a function \( x \rightarrow A(x) \) from \( \mathbb{R}^4 \) into \( \mathcal{L}(\mathcal{H}) \) such that
\[
A(x) = U(x, 1) A(0) U(x, 1)^{-1}
\]
and \( A(0) \in \mathcal{A}(\mathcal{O}) \) for some bounded open \( \mathcal{O} \).

3. The representation \( U \) is reducible. In particular there are four closed subspaces \( \mathcal{H}_j \) of \( \mathcal{H} \), with projectors \( E_j(1 \leq j \leq 4) \), invariant under \( U \), such that the restriction of \( U \) to \( \mathcal{H}_j \) is irreducible, with mass \( m_j > 0 \) and spin zero. \( \mathcal{H}_j \) is associated with a neutral2 stable particle labelled \( j(1 \leq j \leq 4) \). We assume that there are four local ARAKI-HAAG fields \( \{A_j\}_{1 \leq j \leq 4} \) such that:

\[
(\Omega, A_j(0) \Omega) = 0; \quad E_j A_j(0) \Omega \neq 0; \quad (1 - E_j) A_j(0) \Omega
\]
has a mass spectrum \( \geq M_j > m_j \).

Remark. Note that these conditions imply that the states of the form
\[
E_j \int \psi(a, A) U(a, A) A_j(0) \Omega \, da \, dA
\]
\footnote{Actually, it would be sufficient to ascribe an algebra of local observables to each element of a collection \( \mathcal{C} \) of open sets such that: \( \mathcal{O} \in \mathcal{C} \Rightarrow a + A \mathcal{O} \in \mathcal{C} \) for all \((a, A) \in \mathcal{P}^+_4\) containing some bounded open sets.}
\footnote{We consider neutral spinless particles for simplicity, but the generalization to arbitrary charge and spin offers no difficulty.}