The Relativistic Polaron without Cutoffs

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Abstract. The (non-Lorentz covariant) system consisting of a relativistic scalar Boson field $\phi$ interacting with a single spinless particle (relativistic polaron) with kinetic energy function $(m^2 + |p|^2)^{1/2}$ is studied in $d$ space dimensions, where $d \geq 3$. The interaction Hamiltonian is taken to be $\int \Psi(x)^* \Psi(x) \phi(x) \, dx$ where $\phi$ has a momentum cutoff. The physical one polaron Hilbert space $H_{ph}$ for this model, corresponding to no cutoff on $\phi$, is constructed. The total renormalized Hamiltonian $H$ without cutoff is constructed as a semibounded self-adjoint operator on $H_{ph}$. The time zero physical Boson field is also constructed. First order estimates are established for the local (in momentum space) number operators in terms of $H$.

1. Introduction

We consider a single polaron (that is, a spinless electron) interacting with a relativistic scalar Boson field in $d$ space dimensions. We shall be concerned primarily with the cases $d \geq 3$. We take the kinetic energy function of the polaron to be

$$E(q) = (m^2 + |q|^2)^{1/2}$$

where $m$ is the bare mass of the polaron. The total Hamiltonian of the system in the presence of a momentum cutoff $f$ is

$$H_f = H_0 + H_f(f)$$

where

$$H_0 = \int_{E_d} \Psi(p)^* E(p) \Psi(p) \, dp + \int_{E_d} a(k)^* \mu(k) a(k) \, dk$$

$$H_f(f) = \int_{E_d} \{ \Psi(p+k)^* \Psi(p) a(k) + \Psi(p-k)^* \Psi(p) a(k)^* \} f(k) \, dk \, dp$$

$$\mu(k) = (\mu_0^2 + |k|^2)^{1/2}, \mu_0 > 0$$

and $\Psi(p)$ and $a(k)$ are the annihilation operators for the polaron and Boson field respectively. $E_d$ is $d$-dimensional Euclidean space. Our objective is to construct the one polaron physical Hilbert space $H_{ph}$ associated with this model and to show that the total Hamiltonian

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without cutoff (i.e. with \( f(k) = \lambda \mu(k)^{-1/2} \)) can be defined on this space as a non-trivial semi-bounded self-adjoint operator with lower bound equal to the physical mass \( m_0 \) of the polaron. We also construct the time zero Boson fields as operators on \( \mathcal{H}_{ph} \). This model is well suited for studying the phenomenon of infinite field strength renormalization.

We outline the procedure here. Let \( f_n \) denote a sequence of smooth cutoff functions with compact support. We shall allow \( f_n \) to converge in a suitable sense to \( \lambda \mu(k)^{-1/2} \) where \( \lambda \) is a real constant. Write \( H_n \) for the corresponding total Hamiltonian as given by (1.2). We chose the bare polaron mass \( m \) so that the lower bound of \( H_n \) is exactly the physical mass, \( m_0 \) (a given positive constant), of the polaron. As \( n \to \infty \) the bare mass \( m \) (which depends on \( n \)) will go to infinity when \( d \geq 2 \). Thus the model exhibits infinite mass renormalization when \( d \geq 2 \). The bare Hilbert space for the system is

\[
\mathcal{H} = L^2(E_d) \otimes \mathcal{F}
\]

(1.5)

where \( \mathcal{F} \) is the Fock space for the Boson field and \( E_d \) is momentum space for the polaron. Since the theory is translation invariant \( \mathcal{H} \) decomposes into a direct integral of (infinitesimal) subspaces \( \mathcal{F}_p \), on which the total momentum of the polaron plus Boson field has the constant value \( p \), and which reduce \( H_n \). Denoting by \( H_n(p) \) the restriction of \( H_n \) to \( \mathcal{F}_p \) we let \( \psi_n \) be the unique lowest proper vector for \( H_n(0) \) in \( \mathcal{F}_0 \). The existence and uniqueness of \( \psi_n \) was shown in [8, Theorem 8]. We take \( \psi_n \) to be a unit vector. \( \psi_n \) is the physical rest state of the system with cutoff \( f_n \) and the corresponding eigenvalue of \( H_n(0) \) is \( m_0 \).

When \( d = 2 \) one expects, on the basis of perturbation theory, as well as on experience with the external source model [which corresponds to taking \( E(p) = m \)], and as well on the basis of the result of Nelson [10] for a related model, that \( H_n \) and \( \psi_n \) should converge in some reasonable sense in \( \mathcal{H} \). It has been shown by Sloan [12] that when \( d = 2 \) the vectors \( \psi_n \) lie, in fact, in a norm compact subset of \( \mathcal{F}_0 \) and that there is a subsequence \( n_j \) such that \( \psi_{n_j} \) converges in norm while \( H_{n_j} \) converges in the sense of generalized strong convergence [9, Chapter 8] to a semi-bounded self-adjoint operator \( H \) on \( \mathcal{H} \). His methods do not involve the use of dressing transformations and are therefore quite distinct from those of [10].

When \( d \geq 3 \) perturbation theory indicates that \( \psi_n \) converges weakly to zero. Thus in three or more space dimensions, the model, in all likelihood, exhibits infinite polaron field strength renormalization, and we shall assume this in the following discussion. The weak convergence of \( \psi_n \) to zero is sometimes described by saying that \( \psi_n \) moves out of the Fock space \( \mathcal{F}_0 \) and into the physical Hilbert space \( \mathcal{H}_0 \) which is orthogonal to \( \mathcal{F}_0 \). This can be made meaningful in such a way as to give meaning