The Structure of Space-Time Transformations

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Abstract. Let $T$ be a one-to-one mapping of $n$-dimensional space-time $M$ onto itself. If $T$ maps light cones onto light cones and $\dim M \geq 3$, it is shown that $T$ is, up to a scale factor, an inhomogeneous Lorentz transformation. Thus constancy of light velocity alone implies the Lorentz group (up to dilatations). The same holds if $T$ and $T^{-1}$ preserve $(x - y)^2 > 0$. This generalizes Zeeman's Theorem. It is then shown that if $T$ maps lightlike lines onto (arbitrary) straight lines and if $\dim M \geq 3$, then $T$ is linear. The last result can be applied to transformations connecting different reference frames in a relativistic or non-relativistic theory.

1. Introduction

Let $M$ denote $n$-dimensional space-time (Minkowski space). It is an affine space of $n$-tuples $x = (x_0, \ldots, x_{n-1})$, where $x_0 = ct$. We denote by $x^2$ the quadratic form

$$x^2 \equiv x_0^2 - \sum_{i=1}^{n-1} x_i^2.$$  

(1.1)

Zeeman [1] has shown for $\dim M \geq 3$ that a mapping $T$ of $M$ onto $M$ is an orthochronous Lorentz transformation 2 times a dilatation plus a translation if $T$ and $T^{-1}$ preserve the relation

$$\{(y - x)^2 > 0 \text{ and } x_0 < y_0\}$$  

(1.2)

or the relation

$$\{(y - x)^2 = 0 \text{ and } x_0 < y_0\}.$$  

(1.3)

Since the direction of time plays no particular role in quantum field theory and since moreover time reversal is an important symmetry one should drop time order preservation, which, in fact, is quite a strong continuity condition 3. Thus, instead of Eqs. (1.2) or (1.3) we take as

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1 As usual we call two points $x$ and $y$ timelike if $(y - x)^2 > 0$, spacelike if $(y - x)^2 < 0$, and lightlike if $(y - x)^2 = 0$. The light cone in $x$ consists of all $y$ with $(y - x)^2 = 0$.

2 I.e., linear maps of $M$ which preserve the form (1.1) and preserve time orientation.

3 It follows immediately from the preservation of Eq. (1.2) that $T$ is continuous. Indeed, preservation of (1.2) means that $T$ is continuous with respect to the associated order topology which, however, coincides for finite-dimensional spaces with Euclidean topology as already noted in [5].
preserved relations only

\[(x - y)^2 > 0\]  \hspace{1cm} (1.4)
or

\[(x - y)^2 = 0.\]  \hspace{1cm} (1.5)

The latter just expresses constancy of light velocity. In Section 2 we will prove

**Theorem 1.** Let \( \dim M \geq 3 \) and \( T \) be a \( 1-1 \) map of \( M \) onto \( M \). Then \( T \) and \( T^{-1} \) preserve the relation \((x - y)^2 > 0\) if and only if they preserve the relation \((x - y)^2 = 0\). The group of all such maps is generated by

- (i) the full Lorentz group (including time reversal),
- (ii) translations of \( M \),
- (iii) dilations (multiplication by a scalar).

Preservation of \((x - y)^2 = 0\) by \( T \) and \( T^{-1} \) simply means that light cones are mapped onto light cones. Thus constancy of light velocity \( c \) alone implies the Poincaré group\(^4\) (up to dilatations)!

The main statement of Theorem 1 is that \( T \) is affine ("linear"), and one wonders whether linearity also holds in the nonrelativistic case. Of course, \( c \) is no longer constant in all reference frames. If light cones are mapped onto light cones then lightlike lines\(^5\) are mapped onto lightlike lines since any such line is the intersection of two light cones.

In the relativistic as well as nonrelativistic case the worldlines of light rays (photons) should be mapped onto straight lines. It suffices to consider light sources at rest\(^6\) to be able to apply the next result.

**Theorem 2.** Let \( \dim M \geq 3 \), and let \( T \) be a \( 1-1 \) map of \( M \) onto \( M \) which maps lightlike lines onto (arbitrary) straight lines. Then \( T \) is linear.

**Remarks.** (i) Theorem 1 can be derived from Theorem 2 in a straight-forward way. Our proofs of both theorems are, however, more or less independent (Sections 2 and 3). For \( \dim M = 2 \) the theorems do not hold.

(ii) For linearity, the rotational symmetry of the light cone ("isotropy of light velocity") is not crucial. It suffices that its boundary is sufficiently smooth, i.e., sufficiently smooth directional dependence of light velocity.

(iii) Our results severely limit the form of any physical theory which retains straight lines for light rays and which has no distinguished reference frames. For then the laws of Physics have to be form-invariant under transformations connecting different reference frames.

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\(^4\) I.e., the inhomogeneous Lorentz group, generated by (i) and (ii).

\(^5\) A (straight) line is called timelike, spacelike, or lightlike if any two points on it are timelike, spacelike, or lightlike, respectively.

\(^6\) For these the light velocity is also \( c \) in the nonrelativistic case. But in another reference frame the light velocity becomes direction dependent.