The Center-of-Mass in Einsteins Theory of Gravitation

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Abstract. We prove the existence and uniqueness of a center-of-mass line as well as a center-of-motion line, the latter due to G. Dixon, 1964. The validity of the theorems depends on some assumptions listed in § 2, whose most restrictive ones (in the sense of physics) state a certain weakness of the gravitational field. In the concluding paragraph we give some corollaries and a very simple application to the problem of motion.

1. Motivation

To determine the motion of a finite number of material particles one needs three sets of equations: (I) the field equations, (II) the equations of motion and (III) the supplementary equations. By (III) we mean all equations (if necessary inequalities too) that determine the problem uniquely, e.g. equations of state. By (II) one usually means a set of differential equations giving as solution timelike curves in the space time $V^4$ ($V^4$ meant as solution of (I)), such that the curves are uniquely attached to the particles. Up to now it is unknown whether there exist solutions $g_{ab}(x)$ of (I) such that the support of the matter field $T^{ab}(x)$ is a finite set of timelike lines. We therefore take the point of view that $T^{ab}(x)$ means a collection of extended sources (we make it precise in § 2).

Now two problems arise: (a) To find a timelike curve uniquely defined by the given matter distribution; (b) Einsteins classical problem, to derive from (I) the equation of motion (II) for the curves already found in (a). Obviously (b) makes sense only when (a) has been solved. This paper is devoted to the problem (a).

Up to now an answer to (a) was given either by taking over the center-of-mass line of Special Relativity to curved space time, which works for weak fields in connection with a suitable approximation method (Fock, 1939, et al.); or by taking singular sources and taking as the required timelike curve the support of $T^{ab}(x)$ (Einstein, Infeld, Hoffmann, 1938; Taub, 1964; Infeld, Plebanski, 1960 et al.). Taking the
second point of view, for the reason of uniqueness one has to introduce supplementary conditions. Many of them have been proposed but G. Dixon, 1964, was the first, who found an algebraic equation fixing the timelike line without using the equations of motion in its formulation; he explicitly states the logical independence of (a) from (b).

Because of the independence of the spacetime geometry of the material field the center of mass line may be introduced in Special Relativity using mainly $T^{ab} u_b = 0$. If we want to include gravitational phenomena we need the whole set of field equations and not the integrability conditions alone, as will become explicit below. All assumptions restricting the generality of $V^4$ are listed in § 2. Having in mind the use of the center-of-mass line in approximation methods, we demand it to be an extension of the center-of-mass line in Special Relativity; (regarding classical mechanics as the "low velocity"-limit of Special Relativity it is even an extension of the classical center-of-mass concept). We therefore devote § 3 to the problem in Special Relativity; we show two possibilities to define the center-of-mass — one due to Synge, 1935; Møller, 1949, the other is an improvement of the idea of Lanczos, 1929; Papapetrou, 1939; the theorem 3.1 states their equivalence. In the rest of the paper we try to take over both definitions to General Relativity: First we construct a timelike unit vectorfield in a sufficiently large region, containing the particles under consideration, by the condition, that it makes the real valued function (4.4) minimum. With the aid of this unique vectorfield (4.14) defines a mapping of complete normed space of continuous time like curves into itself. This mapping is contractive. By the Banach-fixpoint-theorem there exists one and only one line mapped into itself. We call it the "center-of-mass" line of the particles.

Dixon's condition also fixes a unique "center-of-motion"-line which will be proved essentially by reduction to the proof sketched above. But now, in curved space time, both lines are different in general. Finally, in § 5 we give some properties of the center-of-mass so defined.

2. Basic Assumptions

In this paragraph we list all assumptions needed in the following. In some cases it would be too cumbersome, to give the exact formulation here; we give it in the context below.

The most important quantity will be the matter distribution described by the symmetric tensorfield $T^{ab}(x)$ with the properties:

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2 Dixon did not prove this but he proposed a procedure that can be extended to a proof as has been shown by W. Kündt (Sem. Hamburg 1965).

3 A more detailed outline of the proofs was given at the London Conference 1965 [1].

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