ON THE KORTEWEG-DE VRIES EQUATION

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Dedicated to Hans Lewy and Charles B. Morrey Jr.

Existence, uniqueness, and continuous dependence on the initial data are proved for the local (in time) solution of the (generalized) Korteweg-de Vries equation on the real line, with the initial function \( \phi \) in the Sobolev space of order \( s > 3/2 \) and the solution \( u(t) \) staying in the same space, \( s = \infty \) being included. For the proper KdV equation, existence of global solutions follows if \( s \geq 2 \). The proof is based on the theory of abstract quasilinear evolution equations developed elsewhere.

1. Introduction

The purpose of this paper is to strengthen the results given in the previous paper [1] on the Cauchy problem for the (slightly generalized) KdV equation

\[
\begin{align*}
(1.1) & \quad u_t + u_{xxx} + a(u)u_x = 0, \quad t \geq 0, \quad -\infty < x < \infty, \\
(1.2) & \quad u(0,x) = \phi(x).
\end{align*}
\]

Here all functions are real-valued, and \( a \) is assumed to be \( C^\infty \) (though we do not always need it).

In [1] a general theory of quasilinear equations of evolution was applied to (1.1-2), to deduce existence.

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uniqueness, and continuous dependence on the initial data for the solution, with the assumption that $\varnothing \equiv H^s$ for some $s \geq 3$ and with the solution $u$ obtained in the class

$$u \in C([0,T);H^s) \cap C^1([0,T);H^{s-3}),$$

where $T$ depends on $\|\varnothing\|_s$. (Here $H^s = H^s(-\infty,\infty)$ is the Sobolev space of order $s$ of $L^2$-type, with the norm denoted by $\|\cdot\|_s$.)

Similar results have been obtained, independently, by Bona and others [2,3] for any $s \geq 2$ and any $T > 0$ for the proper KdV equation (in which $a(u) = u$), and by Saut and Temam [4] for $s > 3/2$ and small $T$ except for the continuous dependence in $H^s$. Since the methods used by these authors are quite different from ours, it would be worth while to show that the method used in [1] is also applicable to $s$ smaller than 3. We shall indeed show that $s > 3/2$ suffices for the existence, uniqueness, and continuous dependence in $H^s$ on the initial data for the local solution. We shall also show that global solutions are obtained if there exists a certain global estimate, which is the case with the proper KdV equation. (We note that recently Cohen Murray [5] proved the existence of global solutions for certain discontinuous initial data.)

More precisely, our theorems read.

**THEOREM I.** (a) Let $s > 3/2$ (not necessarily an integer). For any $\varnothing \in H^s$, there is a unique solution $u$ to (1.1-2) in the class (1.3), with $T$ having a lower bound depending only on $\|\varnothing\|_s$.

(b) The map $\varnothing \mapsto u(t)$ is continuous in the $H^s$-norm. More precisely, if $\varnothing_n \in H^s$, $n = 1,2,\ldots$, with $\|\varnothing_n-\varnothing\|_s \to 0$ and $T' < T$, the solution $u_n$ for $u_n(0) = \varnothing_n$ exists on $[0,T']$ for sufficiently large $n$ and $\|u_n(t)-u(t)\|_s \to 0$ uniformly in $t \in [0,T']$. 
