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Small Distance Behaviour in Field Theory and Power Counting

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Abstract. For infinitesimal changes of vertex functions under infinitesimal variation of all renormalized parameters, linear combinations are found such that the net infinitesimal changes of all vertex functions are negligible relative to those functions themselves at large momenta in all orders of renormalized perturbation theory. The resulting linear first order partial differential equations for the asymptotic forms of the vertex functions are, in quantum electrodynamics, solved in terms of one universal function of one variable and one function of one variable for each vertex function whereby, in contrast to the renormalization group treatment of this problem, the universal function is obtained from nonasymptotic considerations. A relation to the breaking of scale invariance in renormalizable theories is described.

Introduction

The small distance behaviour of Green's and vertex functions in renormalizable quantum field theories has been extensively studied in a formal way via the renormalization group [1, 2] and, with equivalent results, some other approaches [8, 9]. Here we offer an alternative approach to the same problem, which appears to be rather more direct. It leads to formulas that are exact and become the usual asymptotic ones upon a, in principle controllable, neglect.

We study the effect of inserting one extra mass vertex, or a generalized mass vertex in the sense of Wilson [3], into all Feynman diagrams for all vertex functions. (Such vertices are defined as those for which the sum of the mass dimensions of the composing fields, a scalar and the electromagnetic field having dimension one, a spinor field dimension three half, a derivative dimension one, is less than four, e.g. two for a scalar mass vertex and three for a spinor mass vertex.) By such insertion, the superficial divergence D of the corresponding Feynman integral is reduced[1] (e.g. by two and by one, respectively, for a scalar and a spinor mass vertex). Reduction of the superficial divergence, however, results in decrease of the large-momentum behaviour by the corresponding power of an overall scale factor[1]. This relation between dimension of

[1] See, e.g., Ref. [2] p. 321; the index $\omega(G)$ we call D. Also ibid., p. 341.

the vertices (i.e., of the terms in the interaction Lagrangian density in the Dyson formula) and large-momentum behaviour is basic in renormalization theory [2] and in many applications (Weinberg sum rules[2], renormalization of theories with broken symmetry [4–7]).

We now consider, however, the insertion of one such vertex as the infinitesimal operation to the insertion of such interaction term into the Lagrangian density. If this type of vertex is already in the Lagrangian, this change of the theory is expressible as a change of the renormalized masses and coupling constants that completely specify the theory.

Thus, the insertion of one such vertex into all diagrams for a vertex function is equivalent to forming some linear combination of the partial derivatives of that vertex function with respect to all renormalized parameters. The coefficients in this linear combination are independent of momenta or of the vertex function selected, and therefore can be determined by calculating the effect of the insertion operation upon the vertex functions of nonnegative superficial divergence degree, which themselves define, at particular momenta, the renormalized parameters of the theory. Once these coefficients have been determined, we have obtained for every vertex functions at all (in this process, fixed) momenta a linear partial differential equation, the inhomogeneous term being the vertex function with that extra vertex inserted into all diagrams, which is expressible in standard fashion by certain integral operations.

If in such partial differential equation the inhomogeneous term is neglected, we obtain a partial differential equation for the asymptotic form of that vertex function. To see this, we recall from the usual discussion of small distance behaviour [1, 2] that asymptotically, all momenta becoming large in the sense $p_i \to \lambda p_i$, $\lambda \to \infty$, the vertex function behaves, in renormalized perturbation theory, like[3] λ^D multiplied by, e.g., in quantum electrodynamics, a double power series in the fine structure constant α and in $\alpha \ln \lambda$, whereby all neglected terms have relative to these one or more factor λ^{-1}. In the partial differential equation described, by power counting, the inhomogeneous term does have [1,3] one or more factor λ^{-1} relative to all the homogeneous terms, such that the asymptotic form does obey the homogeneous linear partial differential equation.

The solution of this equation is trivial and involves an unknown function, e.g., the value the asymptotic form takes for $\lambda = 1$ as a function of the coupling constant. The further discussion of this solution, identical with the one from renormalization group arguments [1, 2], is familiar. What has been achieved, for asymptotic formulas, is only that the uni-

[2] See, e.g., the discussion in Ref. [3].

[3] The present statements hold in general only if no nontrivial partial sum of momenta vanishes, and conform with Weinberg's results [28], as far as these are applicable, cp. Refs. [20, 29].