ON THE VELOCITY OF ELECTROMAGNETIC AND GRAVITATIONAL WAVE FRONTS

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It is shown that, in the space of the general theory of relativity, the velocity of an electromagnetic wave front in a non-inertial frame of reference, and of electromagnetic and gravitational wave fronts in a synchronous frame of reference, is constant and equal to the fundamental velocity. It is noted that the concept of a wave front is meaningful in synchronous frames of reference, of which inertial systems in a Minkowski world are a special case.

1. The characteristic surface \( \omega(x^0, x^i) = \text{const} \) of the Maxwell equations (an electromagnetic wave front) satisfies, in both special and general theories of relativity, the equation [1]

\[
\frac{\partial \omega}{\partial x^0} = 0. \quad (1)
\]

Fok [1] has also shown that the law of propagation of a gravitational wave is the same in the Fok-De Donder [1] harmonic coordinates.

The characteristic surfaces of the Einstein equations were, apparently, first studied by Levi-Civita (see the references cited in [2]). In 1930 he obtained the equation of a surface in the form (1). His work was later simplified by Finzi [2], who again obtained the gravitational wave-front equation (1) and, more important, expressions for the discontinuities of the Einstein tensor in explicit form. It follows, from the coincidence of the Maxwell and Einstein characteristic equations, that a gravitational perturbation propagates with the speed of light. However, they did not investigate the properties of an electromagnetic wave front, in particular its speed of propagation, in the presence of a gravitational field, i.e., in the Riemann space of relativity theory. The quantity interpreted by Finzi [2] as the three-dimensional velocity of a gravitational wave cannot, in fact, serve to denote the velocity of anything in general, since it is not a three-dimensional vector relative to the coordinate transformations [3] which do not alter the frame of reference

\[
x'\beta = x'^\alpha (x^\alpha), \quad \frac{\partial x'^i}{\partial x^0} = 0. \quad (2)
\]

This follows immediately from the expression given by Finzi for the modulus \( \mu \) of this quantity,

\[
\mu = \frac{1}{V(\Delta x^0)^2} \left| \frac{\partial \omega}{\partial x^0} \right|
\]

(the formula is transcribed in the notation of the present article).

The question of discontinuities on the characteristic surface of the Maxwell and Einstein equations was considered by both Einstein and Rosen [4], but their analysis was based on a special metric in a very restricted form. Later, Kompaneets [5] succeeded in somewhat generalizing their metric; nevertheless, in the general case, this question remains open.

In 1957 Pirani [6] introduced an invariant criterion for gravitational radiation using the concept of an observer following the field. (Note that the Pirani criterion is not identical to the Zel'manov-Zakharov formulation [7], since it refers to wave solutions of the third type in the Petrov classification.) The isotropy of the 4-velocity of such an observer (the 4-velocity of the observer in [6] is an eigenvector of the Riemann-Christoffel tensor, and not the energy-momentum tensor as is usual in electrodynamics) indicates the presence of gravitational radiation. Thus Pirani proceeds from the explicit assumption that the gravitational wave velocity is the speed of light, and from the hypothesis of the isotropy of the 4-velocity of light in the general theory of relativity.

Finally, Lichnerowicz [8] also has concluded that the characteristic surfaces of the Maxwell and Einstein equations coincide and represent the isotropic hypersurface of Eq. (1).

As an extension of the above work, we shall investigate here the characteristic surfaces of the Maxwell equations as generalized in the space of the general theory of relativity (and of non-inertial systems in the special theory), and we shall prove that the velocity of an electromagnetic or gravitational wave front is constant and equal to unity in any field of attraction, and also obtain a system of algebraic equations for the discontinuities of the electromagnetic field on the wave front.

2. For completeness, we first adduce a simple and correct proof that the characteristic surface of the Einstein equations obeys Eq. (1). We write the left-hand side of the Einstein equations in the following form:

\[
\Omega^{(\mu)(\nu)} \frac{\partial \xi_{\mu}}{\partial x^i} + \text{(products of lower order)},
\]

where

\[
\Omega^{(\mu)(\nu)} = \frac{1}{2} \left[ g^{\tau \sigma} \left( g_{\sigma i} g^{\mu i} - \frac{1}{2} g_{\sigma i} g^{\mu i} - \frac{1}{2} g_{\sigma i} g^{\mu i} \right) - \frac{1}{2} g_{\sigma i} g^{\mu i} g_{\sigma i} + \frac{1}{2} g_{\sigma i} g^{\mu i} g_{\sigma i} + \frac{1}{4} g^{\nu i} \left( g^{\mu i} g_{\sigma i} + g^{\mu i} g_{\sigma i} \right) + \frac{1}{4} g^{\nu i} \left( g^{\mu i} g_{\sigma i} + g^{\mu i} g_{\sigma i} \right) \right].
\]

From [9], the characteristic surface satisfies

\[
| \Omega^{(\mu)}_{\nu} | = 0, \quad (3)
\]
in which the left-hand side is the determinant of the tenth-order square matrix $\Omega^{(10)}_{ij}$, the elements of which are the corresponding linearly independent components of the fourth-rank tensor $s_{ij}$, where

$$s_{ij} = \delta_{ij} \frac{\partial \omega}{\partial x^i} \frac{\partial \omega}{\partial x^j}.$$ 

Since the tensor $\Omega^{(10)}_{ij}$ is contravariant in the two upper indices and covariant in the two lower, the determinant $|\Omega^{(10)}_{ij}|$ represents an invariant of the general group of coordinate transformations.

Thus the left-hand side of Eq. (3) is simply an invariant homogeneous (relative to the components of the vector $\partial \omega/\partial x^i$) form of twentieth degree, with coefficients constructed exclusively from the metric tensor and the Kronecker and Levi-Civita symbols. It is easy to see, however, that the only quantity from which such a form can be constructed is

$$\Phi = g^{\mu \nu} \frac{\partial \omega}{\partial x^\mu} \frac{\partial \omega}{\partial x^\nu},$$

so that the characteristic Eq. (3) can be rewritten in the form

$$|\Omega^{(10)}_{ij}| = \Phi^{10} = 0,$$

where $\Phi$ is some numerical coefficient. Since $\Phi \neq 0$ (otherwise Eq. (3) would not determine a characteristic surface), Eq. (3) is satisfied by virtue of Eq. (1).

3. In general, it is not possible to solve Eq. (1) when the function $g^{\mu \nu}$ is unknown; therefore, in order to calculate the velocity of an electromagnetic wave front we make use of the method of Levi-Civita, discussed in [10]. When adapted to the purposes of the general theory of relativity, it gives us a completely transparent physical interpretation of the mathematical operations being carried out. The Levi-Civita method does not assume any concrete data on the form and properties of the front, but makes use of the concept and definition of a front and the fact of its existence (and, of course, the field equations). It must be noted that the concept of a front is not simple, but requires considerably more precision. The term "wave front" denotes a two-dimensional surface of simultaneous points in three-dimensional space, on which the field is continuous along the surface but has a weak discontinuity in the direction normal to the surface (along the beam). Operating with the concepts of simultaneity and a nonlocal region of three-dimensional space limits the applicability of the "front" concept to synchronous frames of reference [3]. In this way, the wave front concept which was first invented for inertial frames of reference (which are a subclass of the synchronous frames of reference [3]) is completely defined in synchronous systems, but becomes meaningless in the rest.

Let there exist an electromagnetic wave front $\omega(x^0, x^1, x^2, x^3) = \text{const}$. Fixing the time $t$, we obtain a motionless two-dimensional surface which we can study. We introduce a set of three orthogonal unit vectors $e_1, e_2,$ and $e_3$, with a vertex at the point $M_0(x^0, x^i)$, which satisfy

$$h_{ik} e_i e_j = 1, \quad h_{ik} e_2 e_3 = h_{ik} e_3 e_1 = 0,$$

$$h_{ik} e_1 e_2 = 1, \quad h_{ik} e_2 e_3 = 0, \quad h_{ik} e_3 e_1 = 1,$$

$$e_i = \frac{1}{\sqrt{(\omega(x))^2}} \frac{\partial \omega}{\partial x^i}, \quad (\omega(x))^2 = h_{ik} \frac{\partial \omega}{\partial x^i} \frac{\partial \omega}{\partial x^k}$$

($h_{ik}$ is the metric tensor of the three-dimensional space). The vector $e_1$ is directed along the tangent to the wave beam, and the vectors $e_2$ and $e_3$ lie on the surface. Obviously, this set of vectors has one nonessential degree of freedom, namely the angle of rotation around the axis of the vector normal to the surface, $e_i$.

Now we can expand the derivative of any function $\Psi$ at the point $M_0$ in terms of the unit vectors,

$$\frac{\partial \Psi}{\partial x^i} = \frac{\partial \Psi}{\partial e_1} e_1 + \frac{\partial \Psi}{\partial e_2} e_2 + \frac{\partial \Psi}{\partial e_3} e_3,$$

where $\partial \Psi/\partial e_1$, $\partial \Psi/\partial e_2$, and $\partial \Psi/\partial e_3$ are the derivatives along the corresponding directions. In the neighborhood of the point $M_0$, the function $\Psi$ can be written as a Taylor series,

$$\Psi(x^0 + \Delta x^0, x^i + \Delta x^i) = \Psi(x^0, x^i) + \frac{\partial \Psi}{\partial x^0} \Delta x^0 + \frac{\partial \Psi}{\partial x^i} \Delta x^i + ...$$

If the point $M_1(x^0 + \Delta x^0, x^i + \Delta x^i)$ lies on the wave beam, i.e., $\Delta x^i = e_i \Delta l$, where $\Delta l$ is the spatial distance of $M_0M_1$, then

$$\Psi(M_1) = \Psi(M_0) + \frac{\partial \Psi}{\partial x^i} \Delta x^i + \frac{\partial \Psi}{\partial e_1} \Delta l + ...$$

Let the function $\Psi$ and its first derivatives along $e_2$ and $e_3$ be continuous at the point $M_0$, but let its derivatives along the time coordinate and the direction $e_1$ have discontinuities. We introduce the notation

$$\lim_{\Delta x^0 \to 0} \frac{\partial \Psi}{\partial e_1} \bigg|_{x^0 + \Delta x^0} = \Psi_{x^0},$$

$$\lim_{\Delta x^0 \to 0} \frac{\partial \Psi}{\partial x^i} \bigg|_{x^0 + \Delta x^0} = \Psi_{x^i}.$$