Proof. Since $K$ forms a ring, it follows that for arbitrary $u \in \text{Re } A$ and real number $\lambda$ we have $(\lambda u)^2 = \lambda^2 u^2 \in K \subset \log |A^{-1}|$. In other words, the function $u^2 \in \log |A^{-1}|$ is such that $\mu u^2$ also belongs to $\log |A^{-1}|$ for arbitrary $\mu > 0$. By the lemma we have $u^2 \in \text{Re } A$, i.e., $\text{Re } A$ forms an algebra. By Wermer's theorem we have $A = \mathcal{C}(X)$.

We can prove the following theorem in exactly the same manner by using the appropriate results of Bernard.

**THEOREM 2.** Let $A$ be a uniform algebra on a metrizable compactum $X$. If for an arbitrary function $u \in \text{Re } A$ its modulus $|u|$ belongs to $\log |A^{-1}|$, then $A = \mathcal{C}(X)$. In particular, the function $\varphi(t) = |t|$ can act on $\log |A^{-1}|$ if and only if $A = \mathcal{C}(X)$.

**LITERATURE CITED**


**MODULI OF CONTINUITY OF FUNCTIONS, DEFINED ON A ZERO-DIMENSIONAL GROUP**

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It is shown that the condition $\lim_{|t| \to 0} \omega(t) = \infty$ is a criterion of modulus of continuity in the spaces $C(G)$, $L(G)$, and $L^2(G)$ of functions defined on a zero-dimensional compact Abelian group $G$.

As we know, the criterion of modulus of continuity in the spaces $C(R)$ and $C(T)$ of continuous functions on the line and the circle, respectively, was indicated by Nikol'skii [1] as early as 1946. Besov and Stechkin have found in [2] necessary and sufficient conditions for modulus of continuity in the spaces $L^2(R)$ and $L^2(T)$, which turn out to be essentially different from Nikol'skii's conditions.

As far as we know, criteria of modulus of continuity in $L_p$ for $p \neq 2, \infty$, and also the criteria of higher moduli of continuity for $p \neq 2$ have not yet been established.

In the present note we will establish simple necessary and sufficient conditions for modulus of continuity in the spaces $C(G)$, $L(G)$, and $L^2(G)$ of functions defined on a zero-dimensional compact Abelian group $G$ (see [3] for all the definitions). These conditions are the same for all the indicated spaces (and obviously for arbitrary $L_p(G)$) and are essentially different from Nikol'skii's conditions as well as from the Besov-Stechkin conditions. Besides this, it is observed that a natural generalization of the higher moduli of continuity for zero-dimensional groups is hardly meaningful, since, e.g., for an arbitrary function from $C(G)$ or $L^2(G)$ the second modulus of continuity is estimated from below and from above by the first.
We can regard a zero-dimensional compact Abelian group \( G \) as the set of sequences \( x = (x_1, \ldots, x_n, \ldots) \), in which \( x_n \) assumes the values 0, 1, \ldots, \( p_n-1 \) (\( p_n \) is a prime) with coordinatewise addition modulo \( p_n \) as the group operation and the topology defined by the basic chain of subgroup-neighborhoods of zero

\[
G = U_0 \supset U_1 \supset \ldots \supset U_n \supset \ldots ,
\]

where \( U_n \) is the set of those \( x \) whose first \( n \) coordinates are zero.

With the aid of the system of subgroups (1) we can define (N. Ya. Vilenkin) for an arbitrary complex-valued function \( f(x) \in L_p(G) \) (i.e., integrable in the \( p \)-th power with respect to the Haar measure) its modulus of continuity \( \omega^p(f) \) as the sequence of numbers

\[
\omega_n^p(f) = \sup_{h \in U_n} \|f(x + h) - f(x)\|_{L_p(G)}, \quad 1 \leq p \leq \infty.
\]

(By definition, \( L_\infty(G) = C(G) \) is the space of continuous functions on \( G \).) An equivalent definition is given by Morgenthaler [4].

It is obvious that

\[
\omega_n^{(p)}(f) > \omega_n^{(p-1)}(f) > \ldots > \omega_n^{(1)}(f) = \omega_n(f), \quad \lim \omega_n^{(p)}(f) = 0.
\]

Here the following theorem is proved.

**Theorem.** For an arbitrary sequence \( \omega = \{\omega_n\}_{n=0}^\infty \) there exist functions \( f(x) \in C(G) \), \( f_1(x) \in L_1(G) \), and \( f_2(x) \in L_2(G) \), such that

\[
\omega_n^{(p)}(f) = \omega_n^{(1)}(f_1) = \omega_n^{(1)}(f_2) = \omega_n, \quad n = 0, 1, \ldots
\]

Together with (2) this theorem gives a criterion of modulus of continuity in the spaces \( C(G) \), \( L(G) \), and \( L_2(G) \).

The indicated fact is not quite so unexpected if we take into account the fact that Efimov [5] has established the two-sided estimate

\[
E_{m_n}^{(p)}(f) \leq \omega_n^{(p)}(f) \leq 2E_{m_n}^{(p)}(f), \quad 1 \leq p \leq \infty,
\]

where \( E_{m_n}^{(p)}(f) \) is the best approximation of the function \( f(x), x \in G \), in the metric of \( L_p(G) \) by polynomials of order at most \( m_n - 1 \) with respect to a system of character \( X \) (the definition of the numbers \( m_n \) and the system \( X \) is given below).

It is very simple to prove the theorem in the case of \( C(G) \). It is easily seen that the following function is the desired one:

\[
f(x) = \begin{cases} \omega_n & \text{for } x \in U_n - U_{n+1}, \quad n = 0, 1, \ldots, \\ 0 & \text{for } x = 0. \end{cases}
\]

Indeed, an arbitrary element \( x \in G - U_n \) can be represented in the form \( x = y + h^{(n)} \), where \( h^{(n)} \notin U_n \), and \( y \in U_n \). Therefore, for \( h \in U_n \) the sum \( x + h \) belongs to the same coset \( U_n + h^{(n)} \) to which \( x \) belongs. By virtue of (5) the function \( f(x) \) is constant on any such coset if \( h^{(n)} \neq 0 \). Therefore, \( f(x + h) - f(x) = 0 \) for \( x \in G - U_n; h \in U_n \). On the other hand, \( |f(x + h) - f(x)| \leq \omega_n \), for \( x \) and \( h \) from \( U_n \); moreover, \( f(h) - f(0) = \omega_n \) for \( h \in U_n - U_{n+1} \). These relations prove the equality \( \omega_n(f) = \omega_n \). In a conversation with the author N. Ya. Vilenkin communicated that the example of the function (5) was known to him, but was not published anywhere.

A number of notions are needed for constructing the function \( f_{1,2}(x) \). Let us denote by \( X \) a countable orthonormal (with respect to the Haar measure) system of characters (in the sense of Pontryagin) of the group \( G \), i.e., a system of continuous (in the topology of \( G \)) complex-valued functions \( \chi(x) \) whose moduli are equal to one and which satisfy the functional equation \( \chi(x + y) = \chi(x) \cdot \chi(y) \). The system \( X \) is an Abelian group with respect to the operation of pointwise multiplication. As we know (see, e.g., [6]), \( X \) is the limit of a system of extending subgroups \( X_0 \subset X_1 \subset \ldots \subset X_n \subset \ldots \), where \( X_n \) has the finite order \( m_n = p_1 \cdot p_2 \cdot \ldots \cdot p_n (m_0 = 1) \) and is the annihilator of the subgroup \( U_n \), i.e., the equality \( \chi(x) = 1 \) is valid for arbitrary \( x \in U_n \) and \( \chi \in X_n \). Let us introduce a numbering in \( X \) as follows.

Choosing an arbitrary character in \( X_n - X_{n-1} \), we assign it the number \( m_n - 1 \). For an arbitrary number \( n \geq 1 \), represented in the form

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\]