A Generalization of the Spherical Harmonic Addition Theorem

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Abstract. A generalization of the spherical harmonic addition theorem is proved. The resulting polynomial with four parameters, which corresponds to the Legendre polynomial for the usual spherical harmonic addition theorem, is expressed as four different but equivalent series. Each of them is a finite series of the Gegenbauer polynomials. Thereby the symmetry properties of this polynomial are clarified.

§ 1. Introduction and Summary

In several problems [1] of mathematical physics, there is a need to generalize the spherical harmonic addition theorem. Let us consider the functions of two points \( \Omega \) and \( \Omega' \) on a unit sphere,

\[
B_{L,M}^{L',M'}(\Omega, \Omega') = \sum_{m} C(L' L; m, M - m) Y_{l'm}(\Omega) \ Y_{l'M - m}(\Omega'),
\]

which transform by a rotation as the \((2L + 1)\) components of a spherical tensor [2] of rank \( L \). In (1), \( C(L' L; m, M - m) \) is a Clebsh-Gordan coefficient (CGC). From two vectors we can construct only \( L + 1 \) or \( L \) linear independent spherical tensors or pseudo tensors of rank \( L \). Hence, we can expect and also prove that \( B_{L,M}^{L',M'}(\Omega, \Omega') \) can be expanded in the following forms:

\[
B_{L,M}^{L',M'}(\Omega, \Omega') = \sum_{s = \max(0, L - L')} F_{l'; s}(t) B_{L,M}^{L'-s}(\Omega, \Omega'),
\]

for \( l + l' - L = \) even ,

and

\[
B_{L,M}^{L',M'}(\Omega, \Omega') = \sum_{s = \max(1, L - l' + 1)} F_{l'; s}(t) B_{L,M}^{L'-s+1}(\Omega, \Omega'),
\]

for \( l + l' - L = \) odd .

In Eqs. (2) and (2'),

\[
t = \cos \Theta = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\varphi - \varphi')
\]

with \( \Omega = (\theta, \varphi) \) and \( \Omega' = (\theta', \varphi') \), and \( F_{l'; s}(t) \) is a rotationally invariant function of \( t \).

\(^{1}\) H. Joos used functions slightly different from our \( B_{L,M}^{L'-s+1}(\Omega, \Omega') \) in this case.
The splitting of the two cases for \( l + l' - L = \text{even and odd} \) is due to the space reflection properties of \( B_{L,M}^{l,l'}(\Omega, \Omega') \):

\[
B_{L,M}^{l,l'}(-\Omega, -\Omega') = (-1)^{l+l'} B_{L,M}^{l,l'}(\Omega, \Omega').
\]

For \( L = 0 (= M) \), \( l' \) must equal to \( l \), hence, Eq. (2) reduces to the well-known spherical harmonic addition theorem, since

\[
F_{l,0}^l(t) = (-1)^l (2l + 1)^{1/2} P_l(t) \quad \text{and} \quad B_{0,0}^{0,0}(\Omega, \Omega') = 1/4\pi.
\]

Therefore, Eqs. (2) and (2') just correspond to a generalized addition theorem of the spherical harmonics. Our task is to prove (2) and (2') and obtain and explicit formula for \( F_{l,0}^l(t) \).

This problem was first solved by H. Joos [1] in his study of the representation theory of the inhomogeneous Lorentz group and its applications to the scattering amplitudes for two particles with arbitrary spins. Unfortunately his result for \( F_{l,0}^l(t) \) is rather complicated, a double series in \( t \), and its symmetry properties, which will be discussed in § 2 are also obscure.

We meet the same problems as above in the study of a three-particle system (e.g. the quark model for the baryon) and the application of the Regge pole theory to it. This study will be discussed in a separate paper. In this case we must study the analytical properties of \( F_{l,0}^l(t) \) with respect to \( l, l' \) and \( L \). For this purpose Joos's result is not applicable.

In this paper we shall follow a procedure different from that of Joos, and prove (2) and (2') in § 2 by using the rotation matrices. The expression for \( F_{l,0}^l(t) \) obtained in § 2 is a single series which contains bilinear products of the Gegenbauer polynomials and the CGC. In § 3 we simplify this series and obtain four equivalent expressions for \( F_{l,0}^l(t) \) each of which is a linear combination of the Gegenbauer polynomials. The results in § 3 manifest the symmetry properties of \( F_{l,0}^l(t) \) and also admit us to study the analytical properties of it with respect to \( l, l' \) and \( L \).

In § 4 the orthogonality relations of \( F_{l,0}^l(t) \) are discussed. Appendix A is devoted to the proofs of various addition formulas for the bilinear products of the Gegenbauer polynomials. The symmetry properties of \( F_{l,0}^l(t) \) are also reestablished by using these addition formulas. In Appendix B, we prove the various mathematical formulas which are used in the text.

§ 2. The Proof of the Addition Theorem

In this section we shall prove the generalized addition theorems, (2), in the introduction. We follow the procedure which is the most concise one for the derivation of the usual addition theorem of the spherical harmonics [2]. We evaluate (1) in a rotated coordinate system where the point \( \Omega' \) is on the \( z_1 \)-axis and \( \Omega \) lies in the \( x_1 - z_1 \)-plane, i.e. (\( \Omega' \))