A Convolution Integral for Fourier Transforms on the Group $SL(2, C)$

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Abstract. The Fourier transform of a product of two functions on $SL(2, C)$ is expressed as a convolution integral of the Fourier transforms of its factors. With the help of this convolution integral we present the Fourier transform of a polynomially bounded function as a finite linear combination of analytic delta functionals applied to a continuous function on the real line in an improper sense.

1. Introduction

The scattering amplitude for elastic scattering of elementary particles in forward direction can be described by a set of functions on the homogeneous Lorentz group $SL(2, C)$ which are polynomially bounded and analytic in the real variables of the group $SL(2, C)$. Such presentation of the scattering amplitude allows us to perform a phenomenological analysis of elastic forward scattering in the fashion developed by Toller [1]. Toller’s method is based on harmonic analysis on the group $SL(2, C)$. However, the functions we are dealing with in scattering theory are in general not square integrable, and harmonic analysis in the $L^2$-sense is therefore not applicable. Instead Toller proposes to consider these functions as if they generate linear functionals on certain spaces of test functions. Without following this idea in more detail he assumes that their Fourier transforms are analytic functionals. This means in particular that the functions themselves are written as inverse Fourier transforms in which the integration contour extends over complex values $\omega$, where $\omega$ together with an integer $m$ parametrizes all completely irreducible representations of $SL(2, C)$ [3]. This integral representation which by definition of analytic functionals converges in the sense of the topology of linear functionals is moreover assumed to converge in a standard sense and to have some other “nice” properties.

In this article we study a general approach to polynomially bounded functions and their Fourier transform on $SL(2, C)$. We denote an element of $SL(2, C)$ by

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad \det a = a_{11}a_{22} - a_{12}a_{21} = 1,$$

1 The notations used here have been discussed in more detail in [2].
and introduce the notation $|a|$ for the norm of this element,
\[ |a|^2 = \sum_{i,j} |a_{ij}|^2. \]

We note that $|a|^2 \geq 2$. Any polynomially bounded and continuous function $x(a)$ can be split into two factors
\[ x(a) = |a|^{2\sigma} \cdot x_1(a), \]
such that $x_1(a)$ is square integrable and $\sigma$ is a complex number which can always be chosen a real or even non-negative entire number. We want to express the Fourier transform of $x(a)$ by the Fourier transform of each of the functions $|a|^{2\sigma}$ and $x_1(a)$. We recall that in the case of Fourier transformations on the real line (with respect to the additive group of real numbers, we should add) the transform of a product of two functions can be obtained by convoluting the Fourier transforms of the factors in the standard sense. This approach yields the Fourier transform of a polynomially bounded, once continuously differentiable function in terms of a finite linear combination of finite order derivatives of a continuous function, the derivatives performed in the sense of distributions of course. In this article we try to handle polynomially bounded functions on $SL(2, C)$ in an analogous fashion. The main issue is the construction and investigation of kernels by which to perform the convolutions that we expect from the classical example. We refer to the results on the Fourier transform of the distribution $|a|^{2\sigma}$ as displayed in great detail in [2].

2. Notations

Let $k$ be a triangular matrix,
\[ k = \begin{pmatrix} \lambda^{-1} & \mu \\ 0 & \lambda \end{pmatrix}, \quad \lambda, \mu \text{ complex.} \]

Such triangular matrices form a subgroup $K$ of $SL(2, C)$. One-dimensional unitary representations of $K$ can be characterized by a symbol $\chi = (m, \varrho)$ where $m$ is an integer and $\varrho$ is real,
\[ T^\chi_k = \lambda^{-m \frac{i}{2} + \frac{i\varrho}{2}} \lambda^{-m \frac{i}{2} + \frac{i\varrho}{2}} = |\lambda|^{i\varrho} e^{-i m \arg \lambda}. \]

The right cosets $SL(2, C)/K$ can be parametrized by points $z$ in a compact complex plane. In fact, any element $a \in SL(2, C)$ with $a_{22} \neq 0$ may be decomposed uniquely in the fashion
\[ a = k \zeta, \quad k \in K, \quad \zeta = \begin{pmatrix} 1 \\ 0 \\ z \\ 1 \end{pmatrix}. \]