On the Derivation of the Schroedinger Equation in a Riemannian Manifold

K. R. Parthasarathy

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Abstract. Under certain conditions it is shown that the kinetic part of the dynamical operator of a quantum mechanical system with a Riemannian manifold as configuration space is the Laplace-Beltrami operator.

§ 1. Introduction

In his book on the "Mathematical foundations of quantum mechanics" [2], Mackey raises the problem of characterising the kinetic part of the Schroedinger equation in a Riemannian manifold. The main aim of this paper is to show that under certain conditions the dynamical operator of a quantum mechanical system with a Riemannian manifold as its configuration space has its kinetic part locally unitarily equivalent to the Laplace-Beltrami operator. Since in a Riemannian manifold there need not exist one parameter groups of isometries it seems necessary to characterise the Schroedinger operator without using the notion of momentum. In the case of Euclidean configuration space Mackey obtains the kinetic part of the Schroedinger operator by equating the velocity operator to a constant multiple of the momentum operator. Instead we start from the assumption that the acceleration operator is equal to a constant times the force operator.

In general notations and terminology we follow [2]. Regarding the basic properties of Riemannian manifolds and notations of tensor calculus we refer to [1].

§ 2. Quantum Mechanical Systems with one Degree of Freedom

Let \( R \) denote the real line and \( L^2(R) \) the space of all complex valued functions on \( R \) square integrable with respect to the Lebesgue measure. For any complex valued function \( g \) on \( R \) we shall denote by \( g^{(r)} \) the \( r \)-th. derivative of \( g \). We shall adopt the notation \( g \) for both the function \( g \) as well as multiplication by \( g \). For any two operators \( A \) and \( B \) of \( L^2(R) \) into itself we shall denote by \([A, B]\) the operator \( AB - BA\).

Let \( H \) be the dynamical operator of a quantum mechanical system whose state vectors are unit vectors in \( L^2(R) \). If \( x \) denotes the position operator, then \( i[H, x] \) is the velocity operator and \(-[H, [H, x]]\) is the acceleration operator.
We shall now derive the form of $H$ under the assumption that the acceleration operator is a multiplication operator and an energy equation is satisfied.

**Theorem 2.1.** Let $H$ be a symmetric differential operator with twice differentiable coefficients and

\begin{align*}
\text{a)} & \quad m[H, [H, x]] = -v^{(1)} \\
\text{b)} & \quad m[H, [H, x]^2] = c[H, v],
\end{align*}

where $m$ and $c$ are non-zero constants and $v$ is an infinitely differentiable function. Suppose $v^{(3)}$ does not vanish on a set of positive Lebesgue measure. Then $c = -2$ and $H$ is a second order differential operator which is unitarily equivalent to an operator of the form $h \frac{d^2}{dx^2} + \frac{v}{2hm} + \alpha$ where $h$ and $\alpha$ are constants. The unitary equivalence can be effected through a multiplication operator.

**Proof.** Condition b) implies that

\[ m[H, [H, x]] [H, x] + m[H, [H, x]^2] [H, x] = c[H, v]. \]

By condition a) we have

\[ [H, x] v^{(1)} + v^{(1)} [H, x] = -c[H, v]. \]  \hspace{2cm} (2.1)

Suppose

\[ H = a_n \frac{d^n}{dx^n} + a_{n-1} \frac{d^{n-1}}{dx^{n-1}} + \cdots + a_0 \]

where $a_n \neq 0$. By applying the operators on either side of (2.1) to $C^\infty$ functions with compact supports and equating the coefficients of $\frac{d^k}{dx^k}$, $0 \leq k \leq n - 1$, we have

\[ (k + 1) a_{k+1} v^{(1)} + \sum_{r = k + 1}^{n} r a_r \left( \begin{array}{c} r - 1 \\ k \end{array} \right) v^{(r-k)} = -c \sum_{r = k + 1}^{n} a_r \left( \begin{array}{c} r \\ k \end{array} \right) v^{(r-k)}. \]

Putting $k = n - 1$, we get

\[ 2n a_n v^{(1)} = -c n a_n v^{(1)}. \]

Since $v^{(1)} \neq 0$, $c = -2$. Putting $k = n - 3$, we have

\[ \binom{n}{3} a_n v^{(3)} = 0. \]

Since $v^{(3)} \neq 0$ and $a_n \neq 0$, we have $\binom{n}{3} = 0$. I.e., $n \leq 2$. In other words $H$ is a second order differential operator.

Suppose

\[ H = a \frac{d^2}{dx^2} + b \frac{d}{dx} + d. \]

Substituting this operator in condition (a), we obtain

\[ m \left\{ 2a a^{(1)} \frac{d^2}{dx^2} + (2a a^{(2)} + 2b a^{(1)}) \frac{d}{dx} + a b^{(1)} + b b^{(1)} - 2a d^{(1)} \right\} = -v^{(1)}. \]