BÄCKLUND TRANSFORMATION OF NONLINEAR EVOLUTION EQUATIONS*

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Abstract

Using Lax pair and looking for the gauge transformation, we obtain the Backlund transformation in another form (sometimes it is called Darboux transformation) for some nonlinear evolution equations. We shall see that it is more convenient to use this form of Backlund transformation.

1) Backlund transformation is a powerful tool for studying nonlinear evolution equations. A Backlund transformation of an evolution equation is a correspondence or relation between two solutions to this equation. Using this relation, we can obtain a new solution from a known solution. For example, the following relations

\[
\begin{align*}
    u_e + u'_e &= -\frac{1}{2\eta} \sin(u - u'), \\
    u_d - u'_d &= \eta \sin(u + u')
\end{align*}
\]  

(\eta is an arbitrary constant) are a Backlund transformation of Sine-Gordon equation

\[ u_{et} = \sin u. \]  

In fact, it is not difficult to check that if \( u \) and \( u' \) satisfy (1) then \( u' \) is a solution of (2) if and only if \( u \) is a solution of (2) too. In particular, for a trial solution \( u = 0 \), using (1), we obtain another solution of (2):

\[ u = \arctg \exp\left(-\frac{1}{2\eta} x + 2\eta t\right). \]

Usually, to solve (1) is easier than to solve (2) directly, because equation (1) is a completely integrable system of partial differential equations of first order; essentially, it is an ordinary differential equation. Of course, if the form of the solution \( u \) is more complicated, to solve (1) is not so easy too. As we know, there is much discussion about Backlund transformation of nonlinear evolution equations. In this paper, using the Lax pair, we shall give another form of Backlund transformation and illustrate that it is more convenient to use this form of Backlund transformation.

2) As we know, some nonlinear evolution equations can be considered as a completely integrable condition of a system of linear partial differential equations

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of first order, i.e., for a nonlinear evolution equation, there are $2 \times 2$ matrix functions $M$ and $N$, such that the integrable condition

$$\phi_{st} = \phi_{et}$$

of the system of partial differential equations

$$\begin{align*}
\phi_s &= M \phi, \\
\phi_t &= N \phi,
\end{align*}$$

i.e.

$$N_s - M_t - [M, N] = 0$$

(where $[M, N] = MN - NM$) is the given evolution equation. (3) is called the Lax pair of this equation.

The above content can be described in the language of Lie groups. We assume $G$ is a 2-unimodular group (or a 2-linear group) whose fundamental differential form is

$$\Omega = da \cdot a^{-1}, \quad a \in G,$$

i.e.

$$da = \Omega a$$

(It is called the moving equation of the group). From $0 = d(da) = d(\Omega a)$, we obtain the structure equation of group:

$$d\Omega - \Omega \wedge \Omega = 0.$$ 

The above fact can be explained as follows. For a suitable differential form

$$\Omega = M dx + N dt$$

(taking value in the Lie algebra of $G$), the nonlinear evolution equation is equivalent to the structure equation

$$d\Omega - \Omega \wedge \Omega = 0,$$

and the moving equation

$$da = \Omega a$$

is the Lax pair.

**Example.** If

$$\Omega = \begin{pmatrix}
\eta & u \\
-1 & -\eta
\end{pmatrix} dx + \begin{pmatrix}
-2\mu - 4\eta^2 & -u_s - 2\mu u - 4\eta^2 u - 2u^2 \\
2u + 4\eta^2 & 2\eta u + 4\eta^2 + u_s
\end{pmatrix} dt$$

($\eta$ is an arbitrary constant), then $d\Omega - \Omega \wedge \Omega = 0$ if and only if

$$u_t + 6uu_s + u_{sss} = 0$$

(KdV equation).

If

$$\Omega = \begin{pmatrix}
\eta & -u_s^2/2 \\
u_s^2/2 & -\eta
\end{pmatrix} dx + \frac{1}{4\eta} \begin{pmatrix}
\cos u & \sin u \\
\sin u & -\cos u
\end{pmatrix} dt$$

($\eta$ is an arbitrary constant), then $d\Omega - \Omega \wedge \Omega = 0$ if and only if

$$u_{st} = \sin u$$

(Sine–Gordon equation).

If

$$\Omega = \begin{pmatrix}
\eta & u \\
-u & -\eta
\end{pmatrix} dx + \begin{pmatrix}
4\eta^2 + 2\eta u & -u_s - 2\mu u - 4\eta^2 u - 2u^3 \\
u_{ss} - 2\mu u + 4\eta^2 u + 2u^3 & -4\eta^2 - 2\mu
\end{pmatrix} dt$$