RIGIDITY OF COMPACT MINIMAL SUBMANIFOLDS IN A UNIT SPHERE

ABSTRACT. Let $M$ be an $n$-dimensional compact minimal submanifold of a unit sphere $S^{n+p}(p \geq 2)$, and let $S$ be a square of the length of the second fundamental form. If $S \leq \frac{3}{2}n$ everywhere on $M$, then $M$ must be totally geodesic or a Veronese surface.

1. INTRODUCTION

Let $M$ be a smooth compact $n$-dimensional Riemannian manifold immersed minimally in a unit sphere $S^{n+p}$ of dimension $n + p$, and let $S$ be a square of the length of the second fundamental form. By the equation of Gauss, $S = n(n - 1) - R$, where $R$ is the scalar curvature of $M$. Then $S$ is an intrinsic invariant of $M$. Simons [1] proved that if $S \leq n/(2 - 1/p)$ everywhere on $M$, then either $S \equiv 0$ or $S \equiv n/(2 - 1/p)$. Later, Chern et al. [2] determined all minimal submanifolds of $S^{n+p}$ which satisfy $S \equiv n/(2 - 1/p)$. The problem raised in [1] is to estimate the volume for $S$ next to $n/(2 - 1/p)$. As a partial answer to the question, Shen [5] proved if $S \leq n/(1 + \sqrt{(n - 1)/2n})$ everywhere on $M$, then $M$ is either totally geodesic or a Veronese surface in $S^4$.

We improve the pinching constant of Simons and Shen in the case of $p \geq 3$. Namely, we have

**THEOREM.** Let $M$ be a smooth $n$-dimensional compact minimal submanifold of a unit sphere $S^{n+p}(p \geq 2)$. If $S \leq \frac{3}{2}n$ everywhere on $M$, then $M$ must be totally geodesic or a Veronese surface.

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2. PRELIMINARY

Let $M$ be a compact $n$-dimensional minimal submanifold of $S^{n+p}$. We choose a local field of adapted orthonormal frames $\{e_1, e_2, \ldots, e_{n+p}\}$ in $S^{n+p}$, such that $e_1, e_2, \ldots, e_n$ are tangent to $M$. We shall make use of the following convention on the range of indices:

$$1 \leq A, B, C, \ldots \leq n+p, \quad 1 \leq i, j, k, \ldots \leq n, \quad n+1 \leq \alpha, \beta, \gamma, \ldots \leq n+p.$$ 

Let $B = (b_{ij})$ be the second fundamental form of $M$, $S = \|B\|^2 = \Sigma (b_{ij})^2$, and
let $UM_x$ be the unit tangent space of $M$ at point $x$. We define

\begin{equation}
(2.1) \quad f(x) = \max_{u, v \in UM_x} \| B(u, u) - B(v, v) \|^2,
\end{equation}

then $f$ is a measure of an immersion from being totally umbilical. More precisely, we have

**PROPOSITION.** $M$ is totally umbilical (then $M$ is totally geodesic since $M$ is minimal) if and only if $f \equiv 0$.

**Proof.** The proof is an easy calculation.

Let $x \in M$. Suppose

\begin{equation}
(2.2) \quad f(x) = \| B(u_0, u_0) - B(v_0, v_0) \|^2 \neq 0, \quad u_0, v_0 \in UM_x
\end{equation}

then we can choose an adapted frame $\{e_a\}$ at a point $x$ such that

\begin{equation}
(2.3) \quad e_{n+1} = \frac{1}{\sqrt{f(x)}} \left[ B(u_0, u_0) - B(v_0, v_0) \right]
\end{equation}

and the matrix $(h_{ij}^{n+1})$ satisfies

\begin{equation}
(2.4) \quad h_{11}^{n+1} \geq h_{22}^{n+1} \geq \cdots \geq h_{nn}^{n+1}, \quad h_{ij}^{n+1} = o(i \neq j).
\end{equation}

Then with the frame at $x$, if $u_0 = \Sigma_i x^i e_i$, $v_0 = \Sigma_i y^i e_i$, since

\begin{align*}
f(x) &= \left\| \frac{1}{2} \sum_{i,j,a} (x'^i x'^j - y'^i y'^j) h_{ij}^n e_a \right\|^2 \\
&= \left\| \frac{1}{2} \sum_{i,j} (x'^i x'^j - y'^i y'^j) h_{ij}^{n+1} e_{n+1} \right\|^2 \\
&= \left\{ \sum_i [(x'^i)^2 - (y'^i)^2] h_{ii}^{n+1} \right\}^2 \leq (h_{11}^{n+1} - h_{nn}^{n+1})^2,
\end{align*}

we have $f(x) = (h_{11}^{n+1} - h_{nn}^{n+1})^2$, then one may put

\begin{equation}
(2.5) \quad u_0 = e_1, \quad v_0 = e_n.
\end{equation}

In the above frame, since $\{B(e_1, e_1) - B(e_n, e_n)\}$ and $e_{n+1}$ are parallel, we have

\begin{equation}
(2.6) \quad h_{11}^n = h_{nn}^n, \quad \alpha \neq n + 1.
\end{equation}

Taking $e_i, e_j \in T_x M$, and letting $u = (e_i + e_j)/\sqrt{2}$, $v = (e_i - e_j)/\sqrt{2}$, then $u$, $v \in UM_x$, we have

\begin{align*}
(2.7) \quad \sum_2 (h_{ij}^n(x))^2 &= \frac{1}{4} \| B(u + v, u - v) \|^2 \\
&= \frac{1}{4} \| B(u, u) - B(v, v) \|^2 \leq \frac{1}{4} f(x).
\end{align*}