THE BOUNDARY VALUE PROBLEMS FOR THE FILTRATION EQUATION IN PARTIALLY SATURATED POROUS MEDIA*

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Abstract

In this paper we considered the first and the second boundary value problems for one-dimensional filtration in partially saturated porous media. The existence, uniqueness and regularity of weak solutions were obtained. We also studied the interface problems and discussed the asymptotic behavior of weak solutions.

§ 0. Introduction

Consider a one-dimensional flow of fluid through a homogeneous rigid porous medium, such as filtration of water in soil. When the medium remains unsaturated the equation governing the flow is of the form

$$\theta_t = (\theta^m)_{xx} - (\theta^n)_{x}, \quad (n \geq m > 1), \quad (0.1)$$

or

$$\theta_t = (\theta^m)_{xx} \quad (m > 1), \quad (0.2)$$

which is degenerate on the region where $\theta = 0$. This type of equations has been extensively investigated (cf. [2, 3, 6—8, 10]).

In the present paper we shall consider the medium being partially saturated. In this case the flow is governed by the following equation:

$$c(u)_t = u_{xx} - K(u)_x, \quad (0.3)$$

where $u$ is the hydrostatic potential due to capillary suction, $c$ is the volumetric moisture content, $K$ is the hydraulic conductivity and $x$-axis is so chosen that the downward direction is positive. In particular, neglecting the gravitational effect we have

$$c(u)_t = u_{xx}. \quad (0.4)$$

Since $c(u)$ is strictly increasing for $u < 0$ and $c(u) = \theta$, the saturated moisture content, for $u \geq 0$, equation (0.3) and (0.4) are parabolic in the region $u < 0$ but elliptic in the region $u \geq 0$.

Recently some studies have been carried for this case. For the first boundary value problem of (0.4), Fasano and Primicerio[4] proved the local existence and uniqueness in the classical sense, while Van Duyn and Peletier[11] obtained the global existence, uniqueness in weak sense and discussed interface' problems and

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asymptotic behavior for large $t$. Zhou\textsuperscript{12} and An\textsuperscript{13} investigated the Cauchy problem and the second boundary value problem of (0.4), respectively.

This paper aims to study the first and the second boundary value problems of (0.3) under the following hypotheses:

\begin{itemize}
  \item[(H1)] $c(u) \in C^{0+1}(\mathbb{R})$, $c(u) > 0$ is strictly increasing for $u < 0$, equal to 1 for $u = 0$.
  \item[(H2)] $K(u)$ is the same as $c(u)$ except $K(u) = K_0 > 0$ for $u > 0$.
\end{itemize}

In the domain $Q_T = (0, 1) \times (0, T)$ consider the problems

(I) \begin{align*}
  c(u)_t - u_{xx} - K(u) &= v_0(x), & (x, t) \in Q_T, \\
  u(0, t) &= a_0, & u(1, t) &= a_1, \\
  u(x, 0) &= \phi_0(x), & x \in (0, 1),
\end{align*}

(II) \begin{align*}
  u_x(0, t) - K(u(0, t)) &= \delta_0, & u_x(1, t) - K(u(1, t)) &= \delta_1, & t \in (0, T),
\end{align*}

in which $a_0, a_1, \delta_0$ and $\delta_1$ are constants and $v_0(x)$ is a given function satisfying.

\begin{itemize}
  \item[(H3)] There exists $u_0 \in C^{0+1}[0, 1]$ such that $c(u_0) = v_0$ and, for problem (I), it also satisfies $u_0(0) = a_0, u_0(1) = a_1$.
\end{itemize}

In general, (I) and (II) have no classical solution globally. Similar to [11] we define weak solutions of (I) and (II) as follows:

**Definition 0.1.** A function $u(x, t)$ defined on $\bar{Q}_T$ is said to be a weak solution of (I) if

1. $c(u)$ is continuous on $\bar{Q}_T$;
2. $u - \bar{u} \in L^2(0, T; H^1(0, 1))$ where $\bar{u}(x) = (a_1 - a_0)x + a_0$;
3. $u$ satisfies the integral identity

\[
\int_{Q_T} \left[ (u_{xx} - K(u)) \varphi_x - c(u) \varphi_t \right] dx \, dt = \int_0^1 v_0(x) \varphi(x, 0) dx,
\] (0.5)

in which $\varphi \in C^1(\bar{Q}_T)$ and $\varphi(1, t) = \varphi(x, T) = 0$.

**Definition 0.2.** A function $u(x, t)$ defined on $\bar{Q}_T$ is called a weak solution of (II) if

1. $c(u)$ is continuous on $\bar{Q}_T$;
2. $u \in L^2(0, T; H^1(0, 1))$;
3. For all $\varphi \in C^1(\bar{Q}_T)$ satisfying $\varphi(x, T) = 0$,

\[
\int_{Q_T} \left[ (u_{xx} - K(u)) \varphi_x - c(u) \varphi_t \right] dx \, dt = \int_0^1 v_0(x) \varphi(x, 0) dx + \int_0^T \left[ \delta_1 \varphi(1, t) - \delta_0 \varphi(0, t) \right] dt.
\] (0.6)

The sketch of this paper is as follows: Section 1 contains some preliminary works. The existence of weak solutions is proved in section 2. In section 3 the uniqueness and comparison principles are established, while some regularity properties of weak solutions are given in section 4. Section 5 is devoted to discuss interface problems. Finally the behavior of the weak solution of (I) for large $t$ is studied in section 6.