
ATTRACTORS FOR QUASILINEAR SECOND-ORDER PARABOLIC EQUATIONS OF THE GENERAL FORM

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The question of the existence of compact minimal global $B$-attractors and their properties is investigated for quasilinear second-order parabolic equations of the general form in bounded domains under the Dirichlet condition.

In my paper [1], devoted to Navier–Stokes equations, I mentioned that the theory developed in it (it would be natural to call it a global theory of stability) can be applied to many other nonlinear problems for partial differential equations, possessing dissipative properties, including equations (and systems) of parabolic type. In my lectures and communications, delivered at various conferences, I have formulated results on attractors for quasilinear equations of parabolic type and I have outlined the fundamental steps of their proofs; however, so far I have not published any details. Presently, there are several works devoted to attractors for parabolic equations (see survey [2] or the preprint [3], preceding it; they contain the corresponding references). However, as a rule, these works refer to special classes of such equations and, basically, to equations of the form $\nu_t - \nu\Delta \nu + f(\nu) = g(x)$. I have considered, in addition to these equations, the entire class of quasilinear second-order parabolic equations for which the global solvability of the initial–boundary value problem (under some boundary condition) is known and for which all solutions are drawn, sooner or later, into a bounded set. For this class I have proved the existence of a compact minimal global $B$-attractor $M$, having properties similar to the properties of the attractor $\mathcal{M}$ for the Navier–Stokes equations.

Here I present these results for the problem
\[ \begin{align*}
    \varphi_t(x,t) - \sum_{i,j=1}^n a_{ij}(x,\varphi(x,t)) \varphi_{x_i x_j}(x,t) + a(x,\varphi(x,t)) \varphi(x,t) &= 0, \\
    (x,t) &\in \Omega \times \mathbb{R}^+, \\
    \varphi(x,t) &= 0, \quad (x,t) \in \partial \Omega \times \mathbb{R}^+, \\
    \varphi(x,0) &= \psi(x), \quad x \in \Omega,
\end{align*} \]

making use of joint results of Ural'tseva and Ladyzhenskaya regarding its global unique solvability in Hölder spaces, summarized in [4] and [5]. In the sequel it is essential that \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) and that there is no explicit dependence on \( t \). The case of the nonhomogeneous boundary condition \( \varphi(x,t) = \psi(x), \ (x,t) \in \Omega \times \mathbb{R}^+ \), can be reduced in an elementary manner to the case (2).

The definition of the Hölder spaces \( H^r(\Omega), H^{r/2}(\Omega) \) is taken from [4]. From it we also take the notation of the norms \( |\cdot|_{H^r(\Omega)} \) and \( |\cdot|_{H^{r/2}(\Omega)} \) in these spaces. In addition, \( |u|_{H^r(\Omega)} = \max_{x \in \Omega} \|u(x)\| \), while \( |\psi|_{H^{r/2}(\Omega)} = \max_{(x,t) \in \partial \Omega \times \mathbb{R}^+} \|\psi(x,t)\| \). The norm in \( L^p(\Omega) \) will be denoted in the following manner: \( \|\cdot\|_{L^p(\Omega)} \).

We assume that the functions \( a_{ij}, a: M \longrightarrow \mathbb{R} \) forming equation (1), satisfy the following conditions:

I) for \( \forall (x,y,p) \in M \), \( \forall \xi \in \mathbb{R}^n \) we have
\[ \sum_{i,j=1}^n \xi_i a_{ij}(x,\varphi(x,t)) \xi_j \leq \mu_\varphi \sum_{i,j=1}^n \xi_i \xi_j, \]
where \( \mu_\varphi \) and \( \mu_0 \) are some positive constants.

II) \( a_{ij}(x,\varphi(x,t)) \) are differentiable with respect to all of their arguments, the partial derivatives \( \partial a_{ij}/\partial x_k, \partial a_{ij}/\partial \varphi, \partial a_{ij}/\partial p_k \) are bounded on any compactum \( \mathcal{K} \subset M \) and for \( \forall (x,y,p) \in M \), one has the inequalities
\[ \sum_{i,j=1}^n \left| \frac{\partial a_{ij}(x,\varphi(x,t))}{\partial p_k} - \frac{\partial a_{ij}(x,\varphi(x,t))}{\partial p_j} \right| \leq \mu_\varphi (|\varphi| (|p| + 1)^{\beta/2}), \]
\[ \sum_{i,j=1}^n \left| \sum_{k=1}^n \frac{\partial^2 a_{ij}(x,\varphi(x,t))}{\partial x_k \partial p_k} P_k P_k^{\gamma} + \frac{\partial a_{ij}(x,\varphi(x,t))}{\partial \varphi} \right| \leq \mu_\varphi (|\varphi| (|p| + 1)^{\gamma/2+t/2}), \]
\[ \left| a(x,\varphi(x,t)) \right| \leq \mu_\varphi (|\varphi| (|p| + 1)^{\gamma/2+t/2}), \]

in which \( \mu_\varphi: \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) are some continuous nondecreasing functions.

III) The restriction of the function \( a: M \longrightarrow \mathbb{R} \) to any compactum \( \mathcal{K} \subset M \) belongs to the Hölder space \( H^r(\mathcal{K}) \), where \( r \in (0, 1) \) (here \( \alpha \) is the same for all the arguments).

IV) The function \( a: M \longrightarrow \mathbb{R} \) is differentiable with respect to \( \varphi \) and \( p_k \) and the restrictions of \( \partial a/\partial \varphi, \partial a/\partial p_k, \partial a/\partial p_k, \partial a/\partial \varphi \) to any compactum \( \mathcal{K} \subset M \) belong to \( H^r(\mathcal{K}) \).

V) The restrictions of \( a_{ij} \) and \( a \) to any compactum \( \mathcal{K} \subset M \) are elements of \( H^{k+\beta}(\mathcal{K}) \) with \( k \geq 1 \) and \( \beta \in (0, 1) \).

We have the following

THEOREM 1. Assume that \( a_{ij} \) and \( a \) satisfy the conditions I)-III), \( \partial a/\partial \varphi \in C^{\alpha}(\overline{\Omega}) \), while \( \varphi \), occurring in (3), belongs to \( H^{2+\alpha}(\overline{\Omega}) \) with \( \alpha \in (0, 1) \) and satisfies the consistency conditions
\[ \psi \big|_{\partial \Omega} = 0, \quad -a_{ij}(x,\varphi(x)) \varphi_{x_i x_j}(x) + a(x,\varphi(x)) \varphi(x) \big|_{\partial \Omega} = 0. \]