ON AN INTEGRAL TRANSFORM WITH GENERALIZED ASSOCIATED LEGENDRE FUNCTIONS

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We study some new properties of generalized associated Legendre functions of first and second kind $P^{m,n}_k(z)$ and $Q^{m,n}_k(z)$. Applying these functions, we introduce an integral transform that can be used in solving boundary-value problems of mathematical physics.

The generalized associated Legendre functions of first and second kinds $P^{m,n}_k(z)$ and $Q^{m,n}_k(z)$ are linearly independent solutions of the generalized Legendre equation [6]:

$$
\frac{d}{dz} \left[ (1-z^2) \frac{du}{dz} \right] + \left[ (k(k+1) - \frac{m^2}{2(1-z)} - \frac{n^2}{2(1+z)} \right] u = 0,
$$

where $m$, $n$, $k$, and $z$ can in general assume complex values. These solutions can be represented in terms of the Gaussian hypergeometric function [1, 6]:

$$
P^{m,n}_k(z) = \frac{1}{\Gamma(1-m)\Gamma(z-1)^{\frac{m}{2}}} F \left( k - \frac{m-n}{2}, 1 - k - \frac{m-n}{2}; 1 - m; \frac{1-z}{2} \right).
$$

The following restrictions are imposed on $m$, $n$, $k$, and $z$:

$$
k + \frac{m+n}{2} \neq -1, -2, \ldots, \quad k - \frac{m-n}{2} \neq 0, \pm 1, \pm 2, \ldots, \quad m \neq 1, 2, 3, \ldots,
$$

|arg $(z \pm 1)$| $< \pi$, $|z - 1| < 2$,  

and

$$
Q^{m,n}_k(z) = e^{\pi i m} 2^{k - \frac{m-n}{2}} \frac{\Gamma(k + \frac{m+n}{2} + 1)\Gamma(k + \frac{m-n}{2} + 1)}{\Gamma(2k + 2)} \times \frac{(z + 1)^{\frac{3}{2}}}{(z - 1)^{k+\frac{1}{2}+1}} F \left( k - \frac{m-n}{2} + 1, k + \frac{m+n}{2} + 1; 2k + 2; \frac{2}{1-z} \right),
$$

where

$$
k + \frac{m+n}{2} \neq -1, -2, \ldots, \quad k \pm \frac{m-n}{2} \neq 0, \pm 1, \pm 2, \ldots, \quad 2k \neq -2, -3, \ldots, \quad |arg (z \pm 1)| < \pi, \quad |z - 1| > 2.
$$

In the present paper we study certain new properties of the generalized associated Legendre functions $P^{m,n}_k(z)$ and $Q^{m,n}_k(z)$ and use them to introduce an integral transform.

1. Let us study the analytic behavior of $Q^{m,n}_k(z)$.

Lemma 1. For large values of $|z|$, where $z = \cosh \alpha$, $\tau$ is real, and $m$ and $n$ are complex numbers satisfying conditions (5) for existence of the function $Q^{m,n}_k(z)$ the following formula is applicable:

$$
Q^{m,n}_{-\frac{1}{2} + i\tau} (\cosh \alpha) \approx 2^{\frac{n-m}{2} - 1} e^{i(m\pi - \alpha\tau)} e^{-\frac{\pi}{2} \frac{\Gamma\left(\frac{1}{2} + i\tau + \frac{m+n}{2}\right)\Gamma\left(\frac{1}{2} + i\tau + \frac{m-n}{2}\right)}{\Gamma(1+2i\tau)}}.
$$
It follows from (4) that for \( z = \cosh \alpha \) and \( k = -\frac{1}{2} + i\tau \)

\[
Q_{\frac{1}{2}+i\tau}^{m,n}(\cosh \alpha) = e^{\pi im} 2^{m-n-1} \frac{\Gamma \left( \frac{1}{2} + i\tau + \frac{m+n}{2} \right) \Gamma \left( \frac{1}{2} + i\tau + \frac{m-n}{2} \right)}{\Gamma (1+2i\tau)}
\times \frac{(\cos \frac{\alpha}{2})^n}{(\sinh \frac{\alpha}{2})^{n+1+2i\tau}} F\left( \frac{1}{2} + i\tau + \frac{m+n}{2}, \frac{1}{2} + i\tau - \frac{m-n}{2}; 1+2i\tau; \frac{2}{1-\cosh \alpha} \right).
\]

Expressing \( \cos \frac{\alpha}{2} \) and \( \sinh \frac{\alpha}{2} \) in terms of \( e^{\frac{\alpha}{2}} \) and taking account of the fact that \( F \approx 1 \) as \( \alpha \to \infty \), we obtain the required representation.

**Lemma 2.** For large values of \( |\tau| \), where \( \tau \) is real-valued and \( m \) and \( n \) are complex numbers satisfying conditions (5), the following representation holds:

\[
Q_{\frac{1}{2}+i\tau}^{m,n}(\cosh \alpha) = \frac{2^{m-n}}{\sqrt{2\sinh \alpha}} (i\tau)^{m-\frac{1}{2}} e^{i(m\pi - \alpha \tau)} \left[ 1 + O\left( \frac{1}{|\tau|} \right) \right]. \tag{7}
\]

The proof follows from the asymptotic representation of the hypergeometric function \( F(a,b;c;z) \) for large values of \(|\lambda| \) given in Eq. (2.3(16)) of [1]:

\[
F\left( \alpha + \lambda, \alpha - c + 1 + \lambda; \alpha - b + 1 + 2\lambda; \frac{2}{1-z} \right) = \left( \frac{z-1}{2} \right)^{a+\lambda} \lambda^{-\frac{1}{2}}
\times \frac{2^{a+b} \Gamma(\alpha-b+1+2\lambda) \Gamma\left( \frac{1}{2} \right)}{\Gamma(\alpha-c+1+\lambda) \Gamma(\alpha-b+\lambda)} e^{-(a+\lambda)z} (1-e^{-\zeta})^{-c+\frac{1}{2}} (1+e^{-\zeta}) e^{-a-b-\frac{1}{2} z} \left[ 1 + O(\lambda^{-1}) \right],
\]

where \( \zeta \) is defined by the equality \( z \pm \sqrt{z^2-1} = e^{\pm \zeta} \) and \( (1-e^z) = (e^z-1)e^{\mp iz} \); the upper (resp. lower) sign is taken when \( \text{Im } z > 0 \) (resp. \( \text{Im } z < 0 \)).

We set \( z = \cosh \alpha, \lambda = i\tau, \alpha = b = \frac{1}{2} + \frac{m+n}{2}, c = m+1 \). Using formula (2.3(16)) of [1], and the formula for the ratio of two \( \Gamma \)-functions [1, Ch. 1], after simple transformations we obtain the representation (7) from (4).

**Lemma 3.** If \( m \) is an integer satisfying conditions (5), then \( Q_{k}^{m,n}(z) \) has a logarithmic singularity at the point \( z = 1 \), and

\[
Q_{k}^{m,n}(z) \sim -\frac{1}{4} \ln \left| \frac{1}{z-1} \right| P_{k}^{m,n}(z) \left[ 1 + \left( -1 \right)^m \frac{\Gamma(k+m-1) \Gamma\left( -k - \frac{m-n}{2} \right)}{\Gamma(k-m+1) \Gamma\left( -k + \frac{m+n}{2} \right)} \right]. \tag{8}
\]

Formula (8) can be obtained by analytic continuation of the hypergeometric series in formula (4) into the region \(|1-z| < 2 \) [1, Ch. 2] and removing the indeterminacy that arises for integer values of \( m \) using L'Hôpital's rule. The proof is analogous to the one given in [2, Ch. V] for \( Q_{k}^{m-n}(z) \).

**2.** In what follows we shall need conditions under which the Legendre functions \( P_{k}^{m,n}(z) \) and \( Q_{k}^{m,n}(z) \) have no zeros. We do not consider the case of the singular points \( z = \pm 1 \).

When \( k = -\frac{1}{2} \pm i\tau \) and \( m, n, \) and \( \tau \) are real, the function \( P_{-\frac{1}{2}+i\tau}^{m,n}(z) \) has the form

\[
P_{-\frac{1}{2}+i\tau}^{m,n}(z) = \frac{1}{\Gamma(1-m)} \frac{(z+1)^{\frac{\tau}{2}}}{(z-1)^{\frac{\tau}{2}}} F\left( \frac{1}{2} + i\tau - \frac{m-n}{2}, \frac{1}{2} + i\tau - \frac{m-n}{2}; 1-m; \frac{1-z}{2} \right).
\]

For \( m < 1 \) the terms of the hypergeometric series are positive, and in the domain \(|1-z| < 2 \) the function \( P_{-\frac{1}{2}+i\tau}^{m,n}(z) \) has no real zeros.

The method of finding the number of zeros of the Legendre functions of first and second kind with real \( m, n, \) and \( k \) is based on a theorem of Klein [2, Ch. IX]: the hypergeometric function \( F(a,b;c;z) \) with real \( a, b, \) and \( c \) has

\[
E \left[ |a-b| - |c-a-b| - |1-c| + 1 \right] + d
\]