THE MOMENTS OF SPECTRAL DISTRIBUTION FUNCTIONS FOR THE DISTANCES BETWEEN ADJACENT EIGENVALUES OF RANDOM MATRICES

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In this paper we use limit theorems for Borel functions of random variables to find the second moment of the spectral distribution function of the distances between adjacent eigenvalues of a random matrix. Since random matrices are used to study the energy levels of atomic nuclei, the application of the results obtained here in certain fundamental statistical models of heavy atomic nuclei makes it possible to give a good description of experimental data.

Let \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \) be the eigenvalues of a square random symmetric matrix \( \Xi_n \) of order \( n \). We consider the spectral function

\[
\theta_n(x) = (n-1)^{-1} \sum_{i=1}^{n-1} MF(x - (\lambda_i - \lambda_{i+1})),
\]

where \( F(x) \) is the Heaviside step function,

\[
F(x) = \begin{cases} 
1, & x > 0, \\
0, & x \leq 0.
\end{cases}
\]

The random variables \( \xi_i \), equal to the difference between \( \lambda_i \) and \( \lambda_{i+1} \), \( i = 1, n-1 \), are called the spacings of the random matrix \( \Xi_n \), and \( \theta_n(x) \) is their averaged distribution function. We shall assume that the matrix \( \Xi_n \) is such that the eigenvalues \( \lambda_i \) have a distribution density \( q(y_1, \ldots, y_n) \), \( y_1 \geq \cdots \geq y_n \), and the function \( |q(y_1, \ldots, y_n)| \) is symmetric, i.e., invariant under interchange of any two of the variables \( y_i \). This assumption is not too restrictive. We shall prove the following proposition.

Lemma. If the elements \( \xi_{ij} \), \( i \geq j \), \( i, j = 1, n \), of the random matrix \( \Xi_n \) have a joint distribution density \( p(x) \), then the random variables \( \lambda_i \) have a distribution density

\[
q(y_1, \ldots, y_n) = c_n^{-1} \int_{h_{ii} \geq 0, i=1, n} P(HYH')\mu(dH) \prod_{i>j} |y_i - y_j|,
\]

where \( y_1 > \cdots > y_n \), \( c_n = 2^n \pi^{n(n+1)/4} \prod_{i=1}^{n} \Gamma\left(\frac{n-i+1}{2}\right) \),

and the function \( |q(y_1, \ldots, y_n)| \) is symmetric, where \( \mu \) is the normalized Haar measure on the group of orthogonal matrices and \( H = \{h_{ij}\}_{i,j=1, n} \) is an orthogonal \( n \times n \) matrix.

Proof. Formula (2) is proved in [1]. We shall show that \( |q(y_1, \ldots, y_n)| \) is a symmetric function. Assume for definiteness that \( y_1 \) and \( y_2 \) are interchanged. Since

\[
\int_{h_{ii} \geq 0, i=1, n} P(HYH')\mu(dH) = \frac{1}{2^n} \int_{G} P(HYH')\mu(dH),
\]

making the change of variables $H = C\tilde{H}$, where $C = \text{diag} \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 1, \ldots, 1 \right\}$, $\tilde{H} \in G$, and taking account of the fact that $|\prod_{i>j}(y_i - y_j)|$ is symmetric, we verify that the function $|q(y_1, \ldots, y_n)|$ is symmetric. The lemma is now proved.

For the averaged distribution function (1), which describes a random matrix, the following assertion holds [2]:

$$\theta_n(x) = (n - 1)^{-1} \int_0^x \int_{-\infty}^{+\infty} \left( \frac{\partial^2}{\partial u \partial v} \rho(u, v) \right)_{u=y, v=z+y} \, dy \, dz,$$

(3)

where $\rho(u, v)$ is the probability that the eigenvalues lie outside of the interval $(u, v)$: $\rho(u, v) = P\{\lambda_i \notin (u, v), i = 1, n\}$.

The limit of $\theta_n(x)$ as $n \to \infty$ cannot be found in the form (3), because of the presence of the integral over an infinite interval. For that reason we introduce the function

$$\nu_n(x, c) = \frac{1}{2c} \sum_{k=1}^{n-1} F(x - (\lambda_k - \lambda_{k+1}))F(c - |\lambda_{k+1}|).$$

(4)

Upon averaging we obtain

$$M\nu_n(x) = \int_0^x \int_{-c}^{c} (\partial^2 / \partial u \partial v \rho(u, v))_{u=y, v=z+y} \, dy \, dz.$$

We now find the second moment of the function (4).

**Theorem.** Suppose $\rho(u_1, u_2, v_1, v_2) = P\{\lambda_i \notin (u_1, u_2) \cup (v_1, v_2), i = 1, n\}$. Then

$$\lim_{n \to \infty} M\nu_n^2(x_n^{-\frac{1}{2}}, c_n^{-\frac{1}{2}}) = (2c)^{-1}(0.5\pi + q'(x))$$

$$+ 0.5(2c)^{-2} \int_0^x \int_{-c}^{c} \int_{-c}^{c} \frac{\partial^2}{\partial u_1 \partial u_2} \frac{\partial^2}{\partial v_1 \partial v_2} \rho(u_1, u_2, v_1, v_2)_{u_1=y, u_2=z+y, v_1=z, v_2=z+x+y} \, dy \, dz \, dx_1 \, dx_2,$$

(5)

where $q(z) = \prod_{i=1}^{\infty} (1 - \mu_i(z))$ and $\mu_i(z)$ are the eigenvalues of the integral equation

$$\pi^{-1} \int_0^x (x - y)^{-1} \sin(x - y)f(y) \, dy = \lambda f(x), \quad x \in [0, z]$$

(6)

and

$$\rho(u_1, u_2, v_1, v_2) = \left[ M \exp \left\{ \int_{u_1}^{u_2} \eta^2(t) \, dt + \int_{v_1}^{v_2} \eta^2(t) \, dt \right\} \right]^{-2},$$

(7)

$\eta(t)$ is a stationary Gaussian random function, and

$$M\eta(t) = 0, \quad R(t, s) = \pi^{-1} \sin(t - s) / (t - s).$$

**Proof.** It is obvious that

$$M\nu_n^2(x, c) = \sum_{i,j=1}^{n-1} P\{\lambda_i - \lambda_{i+1} < x, \lambda_j - \lambda_{j+1} < x, |\lambda_{i+1}| < c, |\lambda_{j+1}| < c\}.$$