On Eddy Currents in Thin Plates

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Contents: A method of calculation of eddy currents in thin plates is presented. The stationary eddy-current problem is solved by using a scalar potential. The reaction of the eddy currents on the magnetic field is taken into account by application of an integral equation. Further, a combination of finite elements with the integral equation is presented to calculate an approximated solution for optional configuration of the plate. The paper is a continuation of [2].

Über Wirbelströme in dünnen Platten


List of Symbols

- \( B_0 \) magnetic flux density of an external source
- \( e \) unit-vector
- \( F(u) \) functional
- \( g \) thickness of plate
- \( H \) magnetic intensity
- \( J \) eddy-current density
- \( j = \sqrt{-1} \)
- \( k^2 = j \mu_0 \omega \)
- \( M, N \) points in space
- \( r_{MN} \) distance between points \( MN \)
- \( S \) surface of analysed plate
- \( X, Y \) transformation matrices
- \( u \) scalar potential
- \( x, y, z \) Cartesian coordinates
- \( \gamma \) conductivity
- \( \mu_0 \) permeability of free space
- \( \omega \) angular frequency

1 Introduction

Computing eddy currents in three-dimensions remains a major challenge in the numerical calculation of electromagnetic fields. This is the objective of some papers, which have been published in latest issues [1, 2, 3, 5]. These papers are on eddy-current problems where the current is in a conductor whose thickness is small as compared to the skin depth. The current is considered to be on a surface and, in this sense is two-dimensional. Applying this simplification, Nolle [5] and later Albach and Hannakam [1] calculated eddy currents in a thin circular disk and in a flat strip. Finally, Krakowski succeeded to develop an analytical formula for eddy currents in thin circular and rectangular plates. The analytical results have usually a considerable importance, because some general conclusions can be drown. In this case, the conclusion was very surprising: the current distribution in the thin circular plate was similar to the current distribution in a long circular cylinder with an axial magnetic excitation [4]. The first objective of this paper is to prove, that Krakowski [2, 3] solved a slightly different problem than he had approached.

The second objective of the paper is to present a new numerical method of solving eddy-current problems. The method bases upon the scalar potential representation of eddy currents and, this way, it is connected with the procedure used by Krakowski [2, 3].

It is assumed, that we deal with the phasor values of steady state quantities which are sinusoidal in time. The extension to transients, as it is used by other authors [3], is self-evident but is not discussed.
2 Eddy Currents and Their Scalar Representation

The problem is that of computing the eddy-current distribution in a non-magnetic plate of a small thickness, which is placed in an external electromagnetic field characterized by its normal component $B_0$. The plate lies on the $xOy$ plane of the Cartesian coordinate system (Fig. 1). The eddy-currents $J$ generate the secondary magnetic field $H$, whose linkages are defined by Maxwell equations

\[
\nabla \times H = J, \tag{1}
\]
\[
\nabla \times J = -(j\omega u_2 H_x + j\omega B_0) e_z. \tag{2}
\]

Omitting the displacement current is reasonable for quasi-stationary field. Current density in the flat plate can be represent by a scalar function $u(x, y)$ so that

\[
J = \frac{i}{\sigma} \nabla \times u(x, y) e_z. \tag{3}
\]

Hence,

\[
J_x = \frac{1}{\sigma} \frac{\partial u}{\partial y}, \quad J_y = -\frac{1}{\sigma} \frac{\partial u}{\partial x}, \tag{4}
\]

and the continuity of the eddy currents in the plate is ensured. Substituting (3) to (1) one will have a vector equation

\[
\nabla \times \left( H - \frac{i}{\sigma} u(x, y) e_z \right) = 0, \tag{5}
\]

which can be written as a system of partial differential equations:

\[
\frac{\partial H_x}{\partial y} - \frac{\partial H_y}{\partial x} = \frac{1}{\sigma} \frac{\partial u}{\partial y}, \tag{6}
\]
\[
\frac{\partial H_x}{\partial x} - \frac{\partial H_y}{\partial y} = -\frac{1}{\sigma} \frac{\partial u}{\partial x}, \tag{7}
\]
\[
\frac{\partial H_x}{\partial x} - \frac{\partial H_y}{\partial y} = 0. \tag{8}
\]

In that moment Krakowski comes to a wrong conclusion

\[
H_z = \frac{i}{\sigma} u(x, y). \tag{9}
\]

This is possible only when $H_x$ and $H_y$ components are not dependent on $z$. Do we deal with such a case? Obviously, we do not. In section 6 of the discussed paper [2] one can read: "... the normal component of $H_x$ (total field) is continuous, whereas the tangential component is discontinuous on $S$". This is true statement. The magnetic field generated by eddy-currents around a conducting plate is sketched in Fig. 2. Now, Eqs. (6) and (7) can be written as:

\[
\frac{\partial H_x}{\partial y} - \frac{\partial H_y}{\partial x} = \frac{\partial u}{\partial y}, \tag{10}
\]
\[
\frac{\partial H_x}{\partial x} - \frac{\partial H_y}{\partial y} = -\frac{\partial u}{\partial x}, \tag{11}
\]

and considering the small thickness of the plate one can safely assume that almost everywhere:

\[
\frac{\partial H_x}{\partial y} - \frac{\partial H_y}{\partial x} = 2H_y, \tag{12}
\]
\[
\frac{\partial H_x}{\partial x} - \frac{\partial H_y}{\partial y} = 2H_x, \tag{13}
\]

where, $H_x, H_y$ are the tangential components of the secondary magnetic field at the upper surface of the plate. Hence, Eqs. (10), (11) take the form:

\[
H_y = -\frac{1}{2} \frac{\partial u}{\partial y}, \tag{14}
\]
\[
H_x = -\frac{1}{2} \frac{\partial u}{\partial x}. \tag{15}
\]

Equation (8) is satisfied automatically. Comparing Eqs. (4), (14), (15) one will have

\[
2H_y = -J_x \cdot \sigma, \quad 2H_x = J_y \cdot \sigma, \tag{16}
\]

and that is simply the Ampère's circuital law. Recapitulating, the first of the Maxwell's equations (1) can be transformed to a system of two partial differential equations (14), (15). The problem is that the second of the Maxwell's equations (2) contains the normal component of the secondary magnetic