HILBERT SUBSPACES OF THE SPACES $l_p$ HAVING FULL MEASURE

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In the space $l_p$, $1 \leq p \leq 2$, every probability measure is supported on some Hilbert subspace. For $p > 2$ there exist measures, in particular, Gaussian measures in the space $l_p$, for which each Hilbert subspace has measure zero.

Let $(l_p, (\mathcal{U}_p, \mu))$ be a probability space, $1 \leq p < \infty$, where $l_p$ is the Banach space of all real numerical sequences $x = (x_k)$ satisfying the condition

$$\sum_{k=1}^{\infty} |x_k| < \infty$$

with norm $\|x\|_p = \left( \sum_{k=1}^{\infty} |x_k|^p \right)^{1/p}$. $\mathcal{U}_p$ is the $\sigma$-algebra of Borel sets of the space $l_p$, and $\mu$ is a probability measure in the measure space $(l_p, (\mathcal{U}_p))$. We denote by $l_\infty$ the Banach space of all bounded real numerical sequences $x = (x_k)$ with norm

$$\|x\|_\infty = \sup_k |x_k|.$$

A Hilbert space of a normed space $E$ is a linear subspace $H \subset E$ that is the linear hull of some convex, closed, balanced subset $\xi \subset E$ which becomes a Hilbert space when equipped with the norm $\| \cdot \|_H = \inf\{ \lambda \geq 0 : \xi \in \lambda \xi \}$. A subset $\xi$ with this property is called an ellipsoid. In his doctoral dissertation Sudakov [5] proved that any Gaussian measure in a measure space $(l_p, (\mathcal{U}_p))$ ($p \in [1, 2]$) is supported on some Hilbert subspace. He gave a counterexample of a Gaussian measure in the space $l_\infty$ for which every Hilbert subspace has measure zero.

The object of the present paper is:

1. To prove that for $1 \leq p \leq 2$ every probability measure $\mu$ in $l_p$ is supported on some Hilbert subspace.
2. To indicate for $2 < p$ a continuous (non-Gaussian) probability measure $\mu$ for which every Hilbert subspace has $\mu$-measure zero. (In the case of $l_\infty$ we consider this measure on the $\sigma$-algebra generated by cylinder sets.)
3. To indicate for $2 < p < \infty$ a Gaussian measure $\gamma$ in $l_p$ for which assertion 2 holds, i.e., for any Hilbert subspace $H (C, l_p)$, $\gamma(H) = 0$. Before proving these results, we present certain results which we shall subsequently need.

Proposition 1 (the Minlos–Sazonov theorem). In order that a weak distribution $\mu$ extend to a measure in a Hilbert space $H$ it is necessary and sufficient that for any $\varepsilon > 0$ there exist an ellipsoid $\xi \subset H$ of Hilbert–Schmidt type such that for any continuous linear functional $f$

$$\mu(\{ f(\xi) \} > t - \varepsilon),$$

where $\mu_f$ is the distribution of the functional $f$, and $f(\xi)$ is the image of the ellipsoid $(\xi)$ under the mapping $f$.

Proposition 2 (Sudakov [5]). Let $H$ be a Hilbert space, and let $\xi \subset H$ be an ellipsoid such that the sum of the squares of the lengths of its semiaxes is equal to one. Let $f$ be a continuous linear mapping of $H$ onto some Hilbert space $H_1$ under which the image of the unit sphere in $H$ is the unit sphere in $H_1$. Then the image of the ellipsoid $\xi$ is an ellipsoid in $H_1$ having the sum of the squares of its semiaxes not greater than one.

Proposition 3 (Sudakov [5]). Suppose that in the linear space $E$ there is given a Gaussian measure $\gamma$ and that $E \subset E$ is a measurable subspace; then $\gamma(E \mid H) = 0$ or $\gamma(E \mid H) = 1$.

Proposition 4 (Vakhaniya [6]). In the space \( l_p \) \((1 \leq p < \infty)\) equipped with a centered Gaussian measure \( \gamma \), the \( \gamma \)-measure of any ball with positive radius is positive. (Proposition 4 remains valid for any separable Banach space.)

Proposition 5. If \( H \) is a Hilbert subspace of the Banach space \( E \), then the imbedding of \( H \) into \( E \) is topological.

1. The Case \( l_p \), Where \( 1 \leq p \leq 2 \)

**Theorem 1.** Any probability measure \( \mu \) in the space \( l_p \) \((1 \leq p \leq 2)\) is supported on some Hilbert subspace.

We remark first of all that Theorem 1 is obvious for \( p = 2 \), since the space \( l_2 \) is itself a Hilbert space. We must only consider the case \( 1 \leq p < 2 \). We first prove some auxiliary assertions.

**Lemma 1.** Let \( \{a_k\} \) be a sequence of positive numbers, and let

\[
\xi_p(a_k) = \left\{ x = (x_k) : \sum_{k=1}^{\infty} \frac{x_k^p}{a_k^p} \leq 1 \right\}
\]

denote an oriented ellipsoid (i.e., an ellipsoid with principal axes which coincide with the directions defined by the unit vectors of the space \( \mathbb{R}^N \)) having semiaxes of length \( a_k \) in the \( l_p \) norm. If

\[
\sum_{k=1}^{\infty} a_k^{2p - 2p} < \infty,
\]

then the ellipsoid \( \xi_p(a_k) \) is contained in the space \( l_p \).

**Proof.** For the proof it suffices to use the Hölder inequality

\[
\sum_{k=1}^{\infty} |x_k|^p = \sum_{k=1}^{\infty} \left( \frac{|x_k|}{a_k} \right)^p a_k^p \leq \left( \sum_{k=1}^{\infty} a_k a_k^{-p} \right)^{\frac{p}{p'}} \left( \sum_{k=1}^{\infty} \frac{|x_k|^p}{a_k^p} \right)^{\frac{1}{p'}}
\]

where \( \frac{1}{p} + \frac{1}{p'} = 1 \).

If in inequality (3) we set

\[
x = \frac{2}{2 - p} \quad \text{and} \quad s = \frac{2}{p},
\]

then we obtain

\[
\sum_{k=1}^{\infty} |x_k|^p \leq \left( \sum_{k=1}^{\infty} a_k^{2p - 2p} \right)^{\frac{2 - p}{p}} \left( \sum_{k=1}^{\infty} \frac{|x_k|^p}{a_k^p} \right)^{\frac{p}{p'}}.
\]

From the definition (1) of the ellipsoid \( \xi_p(a_k) \) and inequality (2) it follows immediately that in inequality (4) we have \( \sum_{k=1}^{\infty} |x_k|^p < \infty \), i.e., the ellipsoid \( \xi_p(a_k) \) is entirely contained in the space \( l_p \) for \( 1 \leq p < 2 \). The proof of Lemma 1 is complete.

**Remark 1.** Suppose now that \( V_p \) is the unit ball of the space \( l_p \), and suppose that the mapping \( \varphi : l_p \rightarrow V_p(l_p) \) is defined as follows:

\[
\varphi(x) = \begin{cases} x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}
\]

Let \( \nu = \mu \cdot \varphi^{-1} \) be the image of the measure \( \mu \) under the mapping \( \varphi \). It is obvious that \( \nu \) is supported in the unit ball \( V_p \). It is clear that if \( \nu \) is supported on some Hilbert subspace \( H \), then by linearity \( \mu(H) = 1 \). We therefore henceforth assume that the measure \( \mu \) is supported on the unit ball \( V_p \) of our space \( l_p \).

**Lemma 2.** Let

\[
a_k = \left( \int_{V_p} |x_k|^p \, d\mu(x) \right)^{\frac{2-p}{2p}},
\]

where \( \int_{V_p} |x_k|^p \, d\mu(x) \) is the \( p \)-dimensional volume of the unit ball in the space \( l_p \). The function \( \mu \) is well-defined.

918