A NATURAL MODIFICATION OF A RANDOM PROCESS
AND ITS APPLICATION TO STOCHASTIC FUNCTIONAL
SERIES AND GAUSSIAN MEASURES

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It is known that a Gaussian stochastic process can be expanded in a functional series with random
independent coefficients. In the case where the process is continuous in mean but there exists
no modification of it with continuous simple functions, the series does not converge uniformly. In
what cases does it converge pointwise? This question reduces to the well-studied problem of the
boundedness of the sample functions. It is shown that the pointwise convergence of the expansion
mentioned above is equivalent to the continuity of the sample functions of the process in a certain
separable metric. Some other properties of Gaussian processes and measures are considered,
and generalizations to the non-Gaussian case are given.

In the present work we consider the question of how to distinguish the most natural modification among
all separable modifications of a stochastic process. Such a modification is found for each process of a broad
class which includes sums of stochastic functional series with independent terms and, in particular, Gaussian
processes under the condition that the realizations of the process are locally bounded. The results obtained
are new even for Gaussian processes, although the general properties of Gaussian processes with bounded
but discontinuous realizations were studied in the works of Ito and Nisio [1], Jain and Kallianpur [2], and Sudakov
[3]. We note that the "natural modification" is defined below for a stochastic process on an abstract set T;
in contrast to the definition of a separable modification, our definition does not require the presence of a
topology (or any system of subsets) on T. The restriction of a separable modification to a subset \( T \subset T \) may
not be separable (if \( T_1 \) is not open), while the restriction of the natural modification is always natural. Never-
theless, in the majority of cases where on the one hand there exists a modification with certain good properties
and on the other hand there exists a natural modification, then the latter also possess these properties. We
also consider the question of the pointwise convergence of a stochastic functional series in the situation where
uniform convergence may not occur even locally, while convergence with probability 1 does hold but does not
automatically imply pointwise convergence on an uncountable set. The corresponding results are easily re-
formulated in terms of the weak convergence of stochastic series in Banach spaces (generally, nonseparable).
Gaussian measures in nonseparable Banach spaces are given special consideration. It is proved that continu-
ous linear functionals are always measurable with respect to Gaussian measures; contrary to first impression,
this is not obvious and ceases to hold even for measures of the type of a "uniform distribution on an infinite-
dimensional cube."

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began. Some other results closely related to the present work are formulated in [4].

Definitions and Results

The probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) is assumed to be a Lebesgue space in the sense of Rokhlin [15]. The
space of measurable real functions on \( \Omega \) factored in the usual way is denoted by \( \mathcal{S}(\mathbb{P}) \). A stochastic process
on a set \( T \) is a mapping \( \xi : T \rightarrow \mathcal{S}(\mathbb{P}) \). A modification of the process is a mapping \( \xi : \Omega \rightarrow \mathbb{R}^T \) (\( \mathbb{R}^T \) is the
set of all real functions on \( T \)) possessing the property that \( \forall t \in T \) \( \mathbb{P}\{ \omega : \xi(t)(\omega) = \xi(\omega)(t) \} = 1 \). In other
words, we shall never need to distinguish modifications \( \xi_1 \) and \( \xi_2 \) such that \( \mathbb{P}\{ \omega : \xi_1(\omega) = \xi_2(\omega) \} = 1 \). In saying
that \( \xi \) has a realization possessing a certain property we mean that for almost every \( \omega \) the function \( \xi(\omega) \)
defined on \( T \) possesses the property indicated. The expression "\( \xi \) has a bounded realization" means, of
course, that \( \xi \) has a modification with bounded realizations, i.e., the set \( \xi(T) \) is bounded in \( \mathcal{S}(\mathbb{P}) \) (as in a
partially ordered space; other concepts of boundedness of a set in \( \mathcal{S}(\mathbb{P}) \), e.g., the linear topological concept,
are not found in the present work).
Definition 1. A modification \( \xi : \Omega \to \mathbb{R}^T \) is called natural if there exists a metric \( \rho \) on the set \( T \) such that \( (T, \rho) \) is a separable metric space and \( \xi \) has continuous realizations on \( (T, \rho) \).

The following Theorems 1 and 2 are intended to illuminate various aspects of the concept introduced, while Theorems 3-6 indicate conditions for the existence of a natural modification.

**THEOREM 1.** For an arbitrary set \( C \subseteq S(P) \) the following conditions are equivalent:

a) There exists a stochastic process \( \Sigma : T \to S(P) \) having a natural modification and such that \( \Sigma(T) = C \).

b) Any stochastic process \( \Sigma : T \to S(P) \) such that \( \Sigma(T) \subseteq C \) has a natural modification.

c) The joint distribution of all quantities in \( C \) considered as a Baire measure on the space \( \mathbb{R}^C \) can be extended to a regular dense Borel measure \( \mu \) and \( (\mathbb{R}^C, \mu) \) is a Lebesgue space (\( \mu \) is a completion of the measure \( \mu \)).

d) There exists \( \alpha \in S(P) \) such that each element of the set \( C \) can be represented as a continuous function of \( \alpha \).

e) For any \( \varepsilon > 0 \) there exists \( A \in \mathcal{A} \) such that \( \mathbb{P}(A) \geq 1 - \varepsilon \) and the set \( \{ \xi \cap A : \xi \in C \} \) is contained in the space \( L_{\infty}(P) \) and is separable in its metric (\( \chi_A \) is the indicator function of \( A \)).

f) There exists a metric \( \sigma \) on a set \( \Omega_1 \subseteq \Omega_2 \) of probability one such that \( (\Omega_1, \sigma) \) is a separable metric space, \( \mathbb{P} \) is the completion of some Borel measure on \( (\Omega_1, \sigma) \), and each element of the set \( C \) coincides almost everywhere with a continuous function on \( (\Omega_1, \sigma) \).

**THEOREM 2.** Suppose that the process \( \Sigma : T \to S(P) \) has a natural modification \( \xi \); then:

a) Each natural modification of the process \( \Sigma \) coincides with \( \xi \) almost everywhere.

b) For any Borel set \( B \subseteq \mathbb{R}^T \) the set \( \xi^{-1}(B) \) belongs to the \( \sigma \)-algebra \( \mathcal{A} \).

c) If \( T \) is a linear space and the mapping is linear, then \( \xi \) has a linear realization on \( T \) (there is an analogous assertion for other algebraic structures).

d) If a \( \sigma \)-algebra \( \Sigma \) is given on \( T \) and the mapping \( \Sigma \) is measurable from \( (T, \Sigma) \) to \( S(P) \) with the Borel \( \sigma \)-algebra [corresponding to the usual metric on \( S(P) \)], then \( \xi(\omega, t) \) is measurable in both arguments jointly.

e) If \( T \) is a complete metric space (or a locally compact, regular topological space) and \( \Sigma \) is continuous (in probability), then there exists a set \( T_0 \subset T \) of first category in the Baire sense such that \( \xi \) has realizations which are continuous at all points belonging to \( T \setminus T_0 \) (and this set is dense according to Baire's theorem).

f) If \( T \) is a separable metric space and \( \Sigma \) is continuous (in probability), then the modification \( \xi \) is separable.

**THEOREM 3.** A Gaussian stochastic process \( \Sigma \) on the set \( T \) has a natural modification if and only if there exists a function \( b : T \to [1, +\infty) \) such that the process \( \Sigma(t) = \frac{t}{b(t)} \Sigma(t) \) has bounded realizations (in other words, whenever the set \( \Sigma(T) \) is the countable union of GB sets).

**THEOREM 4 (a generalization of Theorem 3).** Let \( \xi_1, \xi_2, \ldots \) be independent random variables with continuous distributions \( f_1, f_2, \ldots \) which are real-valued functions on a set \( T \). We consider the stochastic functional series

\[
\sum_{k=1}^{\infty} \xi_k(\omega) f_k(t).
\]

We assume that for each \( t \in T \) this series converges with probability 1. Then the following conditions are equivalent.

a) The series \((*)\) converges with probability 1 for all \( t \in T \) simultaneously.

\[\text{If } T \text{ has power greater than the continuum (which is, of course, exotic), then we shall not require that } \rho \text{ separate points of the set } T. \text{ Any } \rho \text{-continuous function must take identical values at points } t_1, t_2 \text{ such that } \rho(t_1, t_2) = 0. \text{ We shall henceforth not repeat this convention.}\]