for any $x > 0$ and all sufficiently large $n$, where $A$ is an absolute positive constant. By condition (8) we have $a_n \leq n$ for all sufficiently large $n$ if the constant $\varepsilon$ is chosen sufficiently small. For these values of $n$ we have $\phi(a_n) \leq \phi(n)$, since the function $\phi$ is nondecreasing. Hence, condition (1) is satisfied with the sum $S_k = \sum_{j=1}^{k} X_j$ replaced by $|S_k|$: therefore, by Theorem 1 we obtain

$$\lim \sup \frac{|S_n|}{a_n} \leq \varepsilon$$  \hspace{1cm} (11)$$

In the definition of $a_n$ we may choose the number $\varepsilon$ arbitrarily small, and therefore (9) follows from (11).

We note a corollary of Theorem 2. Suppose that $\{X_n\}$ is a random sequence which is stationary in the
broad sense with $EX_n = 0$. If the series $\sum |X_j|$ converges, then

$$S_n = 0 \left( \frac{1}{n\phi(n)} \log n \right) a.s.$$

for any function $\phi \in \Psi$.

Theorem 4 of the work [3] follows from the last assertion.

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INFORMATION IN A SCHEME WITH ADDITIVE NOISE

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A formula is proved expressing the information contained in a stationary, linearly regular, Gaussian process with an independent additive increment relative to the original unperturbed process.

1. Let $y$ and $z$ be stationary, linearly regular, Gaussian processes with discrete or continuous time and spectral densities (s.d.) $f_y$ and $f_z$, respectively; let

$$x(t) = y(t) + z(t).$$  \hspace{1cm} (1)$$

We shall consider the scheme with the additive "noise" $z$, i.e., we assume that the processes $y$ and $z$ are independent. In order to eliminate trivial complications, we assume that $Ey(t) = Ez(t) \equiv 0$. If a process $u(t)$, $t \in \mathbb{D} \subset \mathbb{R}$, is given, we denote by $H_u$ the complex Hilbert space generated by the quantities $u(t)$, $t \in \mathbb{D}$, with scalar product $(\xi, \eta) = E\xi \overline{\eta}$ ($\xi, \eta \in H_u$) and by $H_\mathbb{D}$ the subspace of the Hilbert space $H_u$ constructed on the basis of the quantities $u(t)$, $t \in \{a,b\} \cap \mathbb{D}$.

We consider the amount of information $J_t = J_t(x,y)$ contained in the quantities $x_{t-a}$ relative to the quantities $y_{t-a}$. It is known that for processes with discrete time

$$I(x,y) \equiv \lim_{t \to \infty} \frac{1}{t} \int_{-t}^{t} \ln \left| \frac{f_x f_y}{f_{x+y}} \right| d\lambda,$$  \hspace{1cm} (2)$$

where $f_{xy}$ is the mutual spectral density of the pair $(x, y)$,

$$R_{xy}(\tau) \overset{\text{def}}{=} E x(t + \tau) y(t) = \int_{-\infty}^{\infty} e^{i\tau \varphi} f_{xy} \, d\varphi.$$  

In our case, that is assuming the independence of the processes $y$ and $z$, $f_{xy} = f_y$. An analogue of formula (2) in the case where the spectrum of the pair $(x, y)$ is rational has been established in [1]. The general case is considered in [2]. Namely, it was established that if at least one of the processes $x$ or $y$ is weakly regular (the process $x$ is weakly regular if $\tau > 0$ for any $t < \infty$), then for processes with continuous time

$$i(x, y) = \frac{i}{2\pi} \int_{-\infty}^{\infty} \ln \frac{f_{xy}}{f_x f_y} \, d\varphi.$$  

(2')

In the present note conditions are found under which there exists the finite limit \[ \lim_{t \to \infty} \left[ \mathcal{J}_x - ti(x, y) \right] \] and the value of this limit is computed.

2. Let $x(n)$ be a stationary, linearly regular, Gaussian sequence with s.d. $f$. We compute the quantity

$$\mathcal{J}_x \overset{\text{def}}{=} \mathcal{J}(\mathcal{X}_{-\infty}^{+}, \mathcal{X}_{0}^{+}),$$

i.e., the amount of information contained in the "past" of the process $x$ with respect to its "future." Let $g$ be an outer function of the space $H^2$ of the interior of the disk [3] such that $|g(z)|^2 = f(z)$ almost everywhere on the unit circle. The possibility of such a factorization of the s.d. $f$ follows from the condition of linear regularity of the process $x$ [4]. The following theorem holds.

**THEOREM 1.** There is the formula

$$\mathcal{J}_x = \frac{1}{2\pi} \int_{\mathbb{R}} \left| \frac{Q(x)}{Q(z)} \right|^2 \, d\varphi = \frac{1}{2} \sum_k |h_k|^2 \ln,$$

where $ds$ is Lebesgue measure in the plane, $h_k$ are the Fourier coefficients of the function $\ln f$, $\ln f(\varphi) = \sum h_k Z^k$ ($\ln f \in L^1$ if the process $x$ is linearly regular).

**Proof.** We denote by $P^-, P^+$ the orthoprojectors in $\mathcal{X}_{-\infty}^{+}$ onto the subspaces $\mathcal{X}_{-\infty}^{+}, \mathcal{X}_{0}^{+}$, respectively. Then [1] since the process $x$ is Gaussian,

$$\mathcal{J}_x = \mathcal{J}(\mathcal{X}_{-\infty}^{+}, \mathcal{X}_{0}^{+}) = -\frac{1}{2} \ln \det \left( E - P^- P^* P^- \right),$$

(4)

where $E$ is the identity operator. Let $\mathcal{X}(n) = \mathcal{X}^n$. It is known [4] that the mapping extends to an isometry of $\mathcal{X}_{-\infty}^{+}$ onto the $L^2$ space on the unit circle constructed on the basis of the measure $fd\lambda$. The latter space we denote by $L^2_{\mathbb{D}}$. Let $H^2_\lambda$, $H^2_f$ be the subspaces of the space $L^2_{\mathbb{D}}$ generated by the functions $\{Z^n, n \leq 0\}, \{Z^n, n \geq 0\}$, respectively, and let $P^-_f, P^+_f$ be the orthoprojectors in $L^2_{\mathbb{D}}$ onto the subspaces $H^2_\lambda, H^2_f$. Since $VH^- = H^2_\lambda, VH^+ = H^2_f$, by the isometric property of the mapping $V$

$$d_f \overset{\text{def}}{=} \det \left( E - P^- P^* P^- \right) = \det \left( E - P^- P^* P^- \right).$$

(5)

It was established in [5] that

$$d_f = \exp \left\{ -\frac{1}{2\pi} \int_{\mathbb{R}} \left| \frac{Q(x)}{Q(z)} \right|^2 \, d\varphi \right\} = \exp \left\{ -\sum_{k=1}^{\infty} |h_k|^2 \ln \right\}.$$

(6)

This and formulas (5) and (4) imply (3). The proof of the theorem is complete.

Let $x$ be a generalized Gaussian stationary process with continuous time [6], i.e., $x$ is a continuous random functional on the space $\Phi$ of infinitely differentiable functions such that

1) the random variables $x(\varphi), \varphi \in \Phi$, are Gaussian;

2) $m_x(\varphi) \overset{\text{def}}{=} E x(\varphi) = m_x (\tau_\varphi \varphi)$;

3) $R_x(\varphi, \varphi) \overset{\text{def}}{=} E x(\varphi) \overline{x(\varphi)} = R_x (\tau_\varphi \varphi, \tau_\varphi \varphi)$, where $\tau_\varphi$ is the shift operator: $[\tau_\varphi \varphi](t) = \varphi(t + s)$.

We assume that the process $x$ is defined only on the subspace $\Phi_n$ of the space $\Phi$ distinguished by the condition of finiteness of the quantities $E|\varphi(x(\varphi))|$, $\varphi \in \Phi_n$. Let $\mathcal{X}$ be the complex Hilbert space generated by the quantities $x(\varphi), \varphi \in \Phi_n$, with scalar product $(\xi, \xi) = E \xi \overline{\xi}$, and let $\mathcal{X}_n$ be the subspace of the space $\mathcal{X}$.