MATROID INTERSECTION ALGORITHMS*

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Let $M_1 = (E, \mathcal{I}_1), M_2 = (E, \mathcal{I}_2)$ be two matroids over the same set of elements $E$, and with families of independent sets $\mathcal{I}_1, \mathcal{I}_2$. A set $I \in \mathcal{I}_1 \cap \mathcal{I}_2$ is said to be an intersection of the matroids $M_1, M_2$. An important problem of combinatorial optimization is that of finding an optimal intersection of $M_1, M_2$. In this paper three matroid intersection algorithms are presented. One algorithm computes an intersection containing a maximum number of elements. The other two algorithms compute intersections which are of maximum total weight, for a given weighting of the elements in $E$. One of these algorithms is “primal-dual”, being based on duality considerations of linear programming, and the other is “primal”. All three algorithms are based on the computation of an “augmenting sequence” of elements, a generalization of the notion of an augmenting path from network flow theory and matching theory. The running time of each algorithm is polynomial in $m$, the number of elements in $E$, and in the running times of subroutines for independence testing in $M_1, M_2$. The algorithms provide constructive proofs of various important theorems of matroid theory, such as the Matroid Intersection Duality Theorem and Edmonds’ Matroid Polyhedral Intersection Theorem.

1. Introduction

Let $M_1 = (E, \mathcal{I}_1), M_2 = (E, \mathcal{I}_2)$ be two matroids over the same set of elements $E$, and with families of independent sets $\mathcal{I}_1, \mathcal{I}_2$. A set $I \in \mathcal{I}_1 \cap \mathcal{I}_2$ is said to be an intersection of the matroids $M_1, M_2$. In this paper, algorithms are presented for the computation of optimal intersections. We distinguish two types of problems: the cardinality intersection problem, in which we seek an intersection containing a maximum number of elements, and the weighted intersection problem, in which we seek an intersection of maximum total weight, with respect to a given weighting of the elements.

One algorithm is presented for the cardinality intersection problem and two algorithms for the weighted intersection problem. One of the algorithms for the weighted intersection problem is “primal-dual”, being based on duality considerations of linear programming, and the other is

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"primal". All three algorithms are based on the computation of an "augmenting sequence" of elements, a generalization of the notion of an augmenting path from network flow theory and matching theory.

The running time of each algorithm is polynomial in \( m \), the number of elements in \( E \), and in the running times of subroutines for independence testing in \( M_1, M_2 \).

The algorithms provide constructive proofs of various important theorems of matroid theory. Among these are the Matroid Intersection Duality Theorem (maximum cardinality of an intersection equals minimum rank of a covering) and Edmonds' Matroid Polyhedral Intersection Theorem (the intersection of two matroid polyhedra creates a polyhedron whose vertices are vertices of both of the original polyhedra).

2. Matroid definitions

We first review some basic definitions of matroid theory. A matroid \( M = (E, \mathcal{I}) \) is a structure in which \( E \) is a finite set of elements and \( \mathcal{I} \) is a family of subsets of \( E \), called independent sets, such that

\[(2.1) \emptyset \in \mathcal{I} \text{ and all proper subsets of a set } I \text{ in } \mathcal{I} \text{ are also in } \mathcal{I}.\]

\[(2.2) \text{ If } I_p, I_{p+1} \text{ are sets in } \mathcal{I} \text{ containing } p, p+1 \text{ elements respectively, then there exists an element } e \in I_{p+1} - I_p \text{ such that } I_p \cup \{e\} \subseteq I.\]

(Hereafter we shall use the notation "\( I + e \)" for "\( I \cup \{e\} \)" and "\( I - e \)" for "\( I - \{e\} \).")

There are a great variety of matroids which have been discussed in the literature: matric, graphic, transversal, binary, etc. We mention here only one type, which will be used as an example to motivate the discussion which follows. Let \( G = (N, A) \) be a graph with node set \( N \) and arc set \( A \). The graphic matroid \( M = (A, \mathcal{I}) \) of \( G \) has the arcs of \( G \) as its elements and as its independent sets all acyclic subsets \( F \subseteq A \), i.e., all "forests" in \( G \).

Some further definitions which are needed are the following. For a given matroid \( M = (E, \mathcal{I}) \), the rank \( r(A) \) of a subset \( A \subseteq E \) is the cardinality of a maximal independent subset of \( A \). (All maximal independent subsets of \( A \) must have equal cardinality.) A subset of \( E \) which is not independent is dependent. A minimal dependent set is called a circuit. (Circuits need not have equal cardinality.) The span of a set \( A \subseteq E \), denoted \( \text{sp}(A) \), is the maximal superset of \( A \) having the same rank as \( A \). A set \( A \) which is equal to its own span, i.e., \( A = \text{sp}(A) \), is said to be a closed set.