POLYHEDRAL ANNEXATION IN MIXED INTEGER AND COMBINATORIAL PROGRAMMING

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Polyhedral annexation is a new approach for generating all valid inequalities in mixed integer and combinatorial programming. These include the facets of the convex hull of feasible integer solutions. The approach is capable of exploiting the characteristics of the feasible solution space in regions both "adjacent to" and "distant from" the linear programming vertex without resorting to specialized notions of group theory, convex analysis or projective geometry. The approach also provides new ways for exploiting the "branching inequalities" of branch and bound.

1. Introduction

This paper provides a new approach, called polyhedral annexation, for generating valid inequalities for mixed integer programming. Polyhedral annexation provides a particularly effective tool for exploiting problem structure. The inequalities it generates take advantage of structure in a completely general way, without requiring any specific form for the constraints defining the feasible (continuous) region. As in the group theoretic approaches, the inequalities are capable of penetrating regions that may be "distant" from the unit hypercube containing the linear programming vertex. At the same time, as in the convex analysis ("outer polar" and "polaroid") approaches, the inequalities profit from local information about the form of the feasible solution space. The inequalities also, however, make fruitful use of relevant constraints that do not affect the local vicinity of the linear programming vertex.

Successively iterated, polyhedral annexation can provide all relevant inequalities for mixed integer programming. These include the inequalities that succeed in converting the mixed integer problem into a linear program. However, the precise range of inequalities that can potentially be generated is less important than the ability to generate inequal-
ities with particular desired properties. Accordingly, results are given for improving a polyhedral cut by a process of sequential implementation. We also show how to obtain "optimal" inequalities — i.e., supports and facets of the convex hull of feasible solutions — by linear programming. The linear programming problem is expressed in a "primal feasible" form that yields a valid inequality at each iteration. New information is generated as it is needed, proceeding through successive improvements until an optimum or a desired stopping point is reached.

Polyhedral annexation is also a useful supplement to branch and bound. The approach provides new branching schemes as well as improved bounds.

On the negative side of the ledger, polyhedral annexation shares a limitation in common with a number of other efforts to take fuller advantage of problem structure — i.e., its strongest forms require ready access to a substantial amount of information from the updated LP tableau. This is extremely inconvenient for the "product form of the inverse" codes often used in present commercial applications. However, the adaptive aspect of the approach makes it possible to respond to trade-offs between the cost of obtaining updated tableau information and the improvement in the resulting inequality.

2. Formulation

The mixed integer programming (MIP) problem will be written

\[
\begin{align*}
\text{maximize } & \quad x_0 = cx, \\
\text{subject to } & \quad Ax = b, \\
& \quad x_i \geq 0, \quad i \in M = \{1, \ldots, m\}, \quad x_i \text{ integer}, \quad i \in I.
\end{align*}
\]

The index sets $M$ and $I$ may or may not be disjoint, and may or may not contain the indexes of all components of the vector $x$. We will represent the updated linear programming (LP) tableau format for this problem as

\[
\begin{align*}
\text{maximize } & \quad x_0 = x_{00} + \sum_{j \in N} x_{0j}(-t_j), \\
& \quad x_i = x_{i0} + \sum_{j \in N} x_{ij}(-t_j) \quad \text{for all components } x_i \text{ of } x,
\end{align*}
\]