RELATIONS BETWEEN THE HECKE RINGS OF THE GROUPS $\text{Sp}_n$ AND $\text{SL}_n$

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Relations between representations of the Hecke rings of the symplectic and special linear groups on the space of Fourier coefficients of Siegel's modular forms of arbitrary genus are studied. The results are applied to the problem of finding relations between the Fourier coefficients of eigenfunctions of Hecke operators and the corresponding eigenvalues.

1. In problems in the theory of Hecke operators on spaces of Siegel's modular forms it frequently happens that various questions involving representations of the Hecke rings of the symplectic group can be reduced to analogous questions for suitable representations of the Hecke rings of the special linear or full linear group, which are usually simpler. In [1] and [2] such a reduction was used to prove Shimura's conjecture on the rationality of Hecke series of the symplectic group. In the present paper we will describe the main steps of an algebraic scheme enabling us, in principle, to effect an analogous reduction in the important problem of studying the action of Hecke operators on the Fourier coefficients of Siegel's modular forms. Detailed proofs and further applications to the theory of zeta-functions will be published in Matematicheskii Sbornik.

2. Let us recall the definition of Hecke rings. Suppose $\Gamma$ is a subgroup and $\mathcal{S}$ a sub-semigroup of some multiplicative group. We call $(\Gamma, \mathcal{S})$ a Hecke pair if $\Gamma \mathcal{S} = \mathcal{S} \Gamma$ and if for each $g \in \mathcal{S}$ the sets $\Gamma \backslash \Gamma g \Gamma$ and $\Gamma g \Gamma / \Gamma$ are finite. For a Hecke pair $(\Gamma, \mathcal{S})$ we denote by $L(\Gamma, \mathcal{S})$ the free module over the field $\mathbb{Q}$ of rational numbers consisting of all finite formal linear combinations with coefficients in $\mathbb{Q}$ of the left cosets $(\Gamma g)$, $g \in \mathcal{S}$. The group $\Gamma$ acts on $L(\Gamma, \mathcal{S})$ by right multiplications: if $g \in \Gamma$ and $x = \sum \alpha_k g_k \in L(\Gamma, \mathcal{S})$, then

$$x \cdot g = \sum \alpha_k g_k g \in L(\Gamma, \mathcal{S}).$$

Let $\mathcal{A}(\Gamma, \mathcal{S})$ denote the submodule of $L(\Gamma, \mathcal{S})$ consisting of all $\Gamma$-invariant elements. Then $\mathcal{A}(\Gamma, \mathcal{S})$ is an associative ring under the multiplication

$$(\Sigma_i a_i (\Gamma g_i)) \cdot (\Sigma_j b_j (\Gamma h_j)) = \Sigma_{i,j} a_i b_j (\Gamma g_i h_j).$$

We call $\mathcal{A}(\Gamma, \mathcal{S})$ the Hecke ring of the pair $(\Gamma, \mathcal{S})$ (over $\mathbb{Q}$). The distinct elements of the form

$$(\Gamma g_i \Gamma) = \Sigma_i' (\Gamma g_i) \quad (g_i \in \mathcal{S}),$$

(1)

where $\Gamma g_i \Gamma = \bar{\gamma}_i \Gamma g_i$ is the partition of the double coset into a union of distinct left cosets, form a basis of the module $\mathcal{A}(\Gamma, \mathcal{S})$ over $\mathbb{Q}$.

3. We will now introduce the Hecke rings in which we are interested. We will use the following notation: $E_n$ is the identity matrix of order $n$, $0 = 0_n$ is the zero matrix of order $n$, $J_n = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}$. $M_k(\mathbb{Z})$ is the ring of square matrices of order $\mathbb{Z}$ over the ring of rational integers. Everywhere below, we fix an arbitrary natural number $n$ and an arbitrary prime $p$. Put

$$
\Gamma = \Gamma^n = S_p(n) = \{M \in M_{2n}(\mathbb{Z}); \, ^tM J_n M = J_n\},
$$

where $^t$ denotes transposition (this is the integral symplectic group of genus $n$):

$$
S = S_p^n = \{M \in M_{2n}(\mathbb{Z}); \, ^tM J_n M = p^\sigma J_n, \, \sigma = 0, 1, \ldots\};
$$

$$
\Gamma_0 = \Gamma_0^n = \{M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma; \, \det M = 0_n\};
$$

$$
S_0 = S_{0,p}^n = \{M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in S; \, \det M = 0_p\};
$$

$$
\Lambda = \Lambda^n = SL_n(\mathbb{Z});
$$

$$
G = G_p^n = \{M \in M_n(\mathbb{Z}); \, \det M = p^\sigma, \, \sigma = 0, 1, \ldots\}.
$$

It is well known from the theory of elementary divisors for the corresponding groups over the ring $\mathbb{Z}$ that the pairs $(\Gamma, S), (\Gamma_0, S_0), (\Lambda, G)$ are Hecke pairs. Let

$$
L = L_p^n = \mathbb{Z}(\Gamma, S), \quad L_0 = L_{0,p}^n = \mathbb{Z}(\Gamma_0, S_0), \quad H = H_p^n = \mathbb{Z}(\Lambda, G)
$$

be the corresponding Hecke rings. From the abstract point of view, the Hecke rings $L$ and $H$ can be constructed in a simple way: they are isomorphic to commutative polynomial rings over $\mathbb{Q}$ in $n+1$ and $n$ variables, respectively (see [3]). To reveal relations between these rings, which reflect relations between their representations on automorphic forms, we realize both of these rings as certain subrings of $L_0$. The following polynomial representation of the ring plays an essential role. Let

$$
\chi = \Sigma_i a_i (\Gamma_0 M_i)
$$

be an arbitrary element of $L_0$. Each representative $M_i$ of a left coset $\Gamma_0 M_i$ can be chosen in "triangular" form:

$$
M_i = \begin{pmatrix} p^{d_{i0}} & * \ldots * \\ 0 & D_i \end{pmatrix}, \quad \text{where} \quad D_i = \begin{pmatrix} p^{d_{i1}} & * \ldots * \\ 0 & p^{d_{i2}} \ldots * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 \ldots p^{d_{in}} \end{pmatrix}.
$$

We now put

$$
\phi(\chi) = \Sigma_i a_i \prod_{j=0}^{n} (x_j p^{-d_{ij}})^{d_{ij}},
$$

where $x_0, \ldots, x_n$ are algebraically independent over $\mathbb{Q}$. The mapping

$$
\phi : L_0 \rightarrow \mathbb{Q}[x_0, \ldots, x_n]
$$

(2)