Further observations are made on the author's earlier paper (Ref. Zh. Mat., 1977, 5A284) dealing with the lattice $\mathcal{H}$ of all subgroups of the full linear group $GL(n,K)$ over a field $K$ that contain the group $D(n,K)$ of diagonal matrices. It is noted, for example, that for an infinite field $K$ all subgroups in $\mathcal{H}$ are algebraic; a subgroup in $\mathcal{H}$ is connected if and only if it is a net subgroup; the lattice of all connected subgroups in $\mathcal{H}$ is isomorphic to the lattice of all marked topologies on $n$ points; any subgroup $\mathcal{H}$ in $\mathcal{H}$ is a semidirect product $\mathcal{H}=A\cdot H_0$ of a maximal connected normal subgroup $H_0$ of $\mathcal{H}$ and a finite group $A$ of permutation matrices.

Suppose $K$ is an arbitrary field, $G=GL(n,K)$ the full linear group of degree $n \geq 2$ over $K$, and $D=D(n,K)$ the subgroup of diagonal matrices. Let $\mathcal{H}$ denote the lattice of all intermediate subgroups $H$, such that $D \leq H \leq G$. It was shown in [1] that this lattice is finite and, if card $K > 3$, does not depend on the field $K$. In the present note we make a number of additional assertions, mainly without proofs, concerning the subgroups $H$ in $\mathcal{H}$. Where it is not too complicated we formulate the results for the full linear group over a simple Artinian (or local) ring. The notation and terminology are taken from [1] (see also [2]).

Suppose $\Lambda$ is a simple ring (not necessarily Artinian). All $D$-nets of a fixed order $n$ in the ring $\Lambda$ form a finite lattice. On the other hand, all topologies that can be
introduced on the segment \( I = (1, \ldots, n) \) of the natural sequence also form a lattice under the inclusion relation. We will denote it by \( \Gamma(n) \). To each topology \( T \in \Gamma(n) \) we can associate a \( D \)-net \( \sigma \) of order \( n \), by putting \( \sigma_{ij} = \Lambda \), if the point \( i \) in \( I \) is contained in the closure \( \overline{I} \) of the point \( j \) (in the topology \( T \)), and \( \sigma_{ij} = (0) \) otherwise. It is easy to see that the mapping \( T \mapsto \sigma \) is bijective.

**THEOREM 1.** Suppose \( n \) is a natural number and \( \Lambda \) an arbitrary simple ring. Then the lattice of all \( D \)-nets in \( \Lambda \) of order \( n \) is isomorphic to the lattice \( \Gamma(n) \) of all topologies on a fixed finite set of \( n \) elements.

If a topology \( T \) satisfies the separation axiom \( (T_0) \) then the corresponding net \( \sigma \) has the following property: if \( i \neq j \) at least one of the ideals \( \sigma_{ij} \) or \( \sigma_{ji} \) is zero. To the discrete topology (the only topology on \( I \) satisfying the separation axiom \( (T_1') \)) corresponds the \( D \)-net with zero ideals off the main diagonal.

A net \( \omega \) over a simple ring \( \Lambda \) is called a \( U \)-net if for each pair of indices \( i \neq j \) at least one of the ideals \( \omega_{ij} \) or \( \omega_{ji} \) is zero. For a \( U \)-net, in particular, all ideals \( \omega_{ii} \), appearing on the main diagonal, are zero. Any \( U \)-net is similar to a triangular \( U \)-net with zero ideals under the main diagonal. The net group \( G(\omega) \) for a \( U \)-net \( \omega \) is unipotent.

To each \( D \)-net \( \sigma \) in a simple ring \( \Lambda \) we can associate a \( U \)-net \( \omega \) by putting \( \omega_{ij} = (0) \) if \( \sigma_{ij} = \sigma_{ji} = \Lambda \) and \( \omega_{ij} = \sigma_{ij} \) otherwise.

**THEOREM 2.** Suppose \( \sigma \) is an arbitrary \( D \)-net of order \( n \) in a simple ring \( \Lambda \). Then the net group \( G(\sigma) \) decomposes into a semidirect product

\[
G(\sigma) = L \cdot G(\omega),
\]

where \( \omega \) is the above-mentioned \( U \)-net corresponding to the \( D \)-net \( \sigma \), \( G(\omega) \) is a unipotent normal subgroup of \( G(\sigma) \) and \( L \) is a group that decomposes into a direct product of full linear groups of smaller orders:

\[
L \simeq GL(\kappa_1, \Lambda) \times \cdots \times GL(\kappa_m, \Lambda), \quad \kappa_1 + \cdots + \kappa_m = n.
\]

The decomposition of \( G(\sigma) \) indicated in Theorem 2 can be viewed as an analog of the Levi decomposition well known in the theory of algebraic groups.

A \( D \)-net \( \sigma \) of order \( n \) over a simple ring \( \Lambda \) defines on the index set \( I = (1, \ldots, n) \) an equivalence relation: \( i \equiv j \) if and only if \( \sigma_{ij} = \sigma_{ji} = \Lambda \). This equivalence relation, in turn, defines a partition of the set \( I \) into equivalence classes:

\[
I = I_1 \cup I_2 \cup \cdots \cup I_m.
\]

For definiteness we number the classes \( I_k \) as follows. Let \( I_1 \) denote the class containing the number 1; then let \( I_2 \) denote the class containing the smallest number not in \( I_1 \); then let \( I_3 \) denote the class containing the smallest number not in \( I_1 \cup I_2 \); and so on.

The symmetric group \( S_n \) on degree \( n \) acts as a group of operators on the nets of order \( n \) as follows:

\[
(\sigma^\tau)_{ij} = \sigma_{\tau(i)\tau(j)}, \quad \tau \in S_n.
\]

Let \( P(\sigma) \) denote the subgroup of those permutations \( \tau \) in \( S_n \) such that \( \sigma^\tau = \sigma \). Any permutation \( \tau \) in \( P(\sigma) \) maps each class \( I_\ell \) bijectively onto some class \( I_\delta \), so that we have