ON AN EXTENSION PROBLEM, GENERALIZED FOURIER ANALYSIS, AND AN ENTROPY FORMULA

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Let \( h(t) \) be an \( n \times n \) matrix valued function on the interval \( |t| \leq \tau \) with summable entries. Let \( \hat{h} \) denote the Fourier transform of \( h \) and let \( e \) denote the \( n \times n \) identity matrix. Necessary and sufficient conditions for the existence of an extension \( u \) of \( h \) to the full line such that \( e-u \) admits either a left or a right canonical factorization and the inverse transform of \( (e-u)^{-1} - e \) vanishes for \( |t| \geq \tau \) are presented and discussed. The connections between these extensions and a generalized Fourier transform are then explored in detail with the help of the theory of triangular factorization. It is then shown that if an allied finite Wiener-Hopf operator based on \( h \) is positive, then \( h \) admits exactly one extension of the type alluded to above. This extension is then characterized in terms of an entropy integral.

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0. **NOTATION**

Throughout this paper we shall let $\mathbb{R}$ denote the real numbers, $\mathbb{C}$ the complex numbers, and $\mathbb{P}_+$ [resp. $\mathbb{P}_-$] the open upper [resp. lower] half of the complex plane. If $\omega$ is a set [resp. a complex number], then $\bar{\omega}$ stands for the closure [resp. complex conjugate] of $\omega$. The symbol $L^p_{n \times m}$ will denote the space of $n \times m$ matrix valued functions on $\mathbb{R}$ with entries belonging to $L^p$. If $1 \leq p < \infty$, then $L^p_{n \times m}$ is a Banach space with respect to the norm

$$
\|f\|_{L^p_{n \times m}} = \left\{ \sum_{i=1}^n \sum_{j=1}^m \|f_{ij}\|_{L^p}^2 \right\}^{1/2}
$$

where $f_{ij}$ denotes the $ij$ entry of $f$. We shall abbreviate this norm by $\| \|$ and the space $L^p_{n \times 1}$ by $L^p$ and shall, if say the interval of integration is $[a,b]$ and not $\mathbb{R}$, indicate this by the notation $L^p_{n \times m}[a,b]$.

The symbols $\hat{f}$ [resp. $f^\vee$] will denote the Fourier transform [resp. inverse Fourier transform]:

$$
\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(t)e^{i\lambda t} dt \quad \text{and} \quad f^\vee(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\lambda)e^{-i\lambda t} d\lambda .
$$

If $A$ is a matrix, then $A^\times$ [resp. $A^\dagger$] will denote the conjugate transpose [resp. transpose (without conjugation)] of $A$, whereas if $A$ is an operator in a Hilbert space, then $A^*$ will denote the operator adjoint.

The symbol $e$ appearing without superscripts (as opposed to say $e^{i\lambda t}$) will always denote the $n \times n$ identity matrix: $\text{diag}(1,1,\ldots,1)$, whereas $I$ will typically denote the identity operator acting in an $L^p_{n \times m}$ space.