A CLASS OF SCATTERING AMPLITUDES WHICH SATISFY THE UNITARITY CONDITION

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The unitarity condition is applied to the Regge representation of the scattering amplitude. This leads to a linear inhomogeneous algebraic equation for the residua of Regge poles and to a linear inhomogeneous integral equation (with Cauchy kernel) for the conical amplitude which can be solved exactly.

INTRODUCTION

It is well known that the combination of the analytical properties of the scattering amplitude and unitarity leads, at least in the case of potential scattering, to a complete dynamical description [1]. However, while the required analytical properties can be expressed simply by means of the appropriate dispersion relation, the unitarity greatly complicates the problem because it leads to the solution of a non-linear integral equation. The only known solution of this equation at present is the scattering amplitude in a Coulomb field. Obviously, by expanding the amplitudes into a series of partial waves one can find infinitely many solutions but these are given in the form of infinite series with Legendre polynomials, as a consequence of which the analytical continuation of the solution to the complete complex plane of the variable cos θ is very difficult. As was shown in [2], expansion with respect to Legendre polynomials permits an analytical continuation in the Lehmann ellipse but when extending the region of analyticity outside the ellipse one must perform the Watson transformation of this expansion. In Regge's representation of the scattering amplitude, which is the result of such a transformation, the analytical properties of the amplitude in the whole complex plane of cos θ are automatically included but the unitarity condition leads here to more complicated relations than in the case of partial waves [3].

In the present paper the author investigates the consequences of the unitarity condition for the Regge representation. It is shown that the unitarity condition can in this case be linearized and the linear equation thus obtained can be solved exactly.

DERIVATION OF AUXILIARY MATHEMATICAL RELATION

Before going on to the actual investigation of the problem let us derive an auxiliary mathematical relation which will be needed later. It will be proved that

$$\int P_s(n_1 \cdot n) P_s'(n_2 \cdot n) \, dΩ_α =$$

$$= \frac{4}{α(α + 1) - α'(α' + 1)} \left[ \sin πα \cdot P_s(-n_1 \cdot n_2) - \sin πα' \cdot P_s(-n_1 \cdot n_2) \right]$$

where $P_s(α)$ is the Legendre function of the first kind with arbitrary (complex) index $α$.
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and the integration relates to the whole solid angle. The proof is based on the equation of definition for $P_\alpha(x)$

$$
\frac{d}{dx} \left[ (1 - x^2) \frac{dP_\alpha(x)}{dx} \right] + \alpha(\alpha + 1) P_\alpha(x) = 0.
$$

The integral on the lefthand side of (1), multiplied by the factor $\alpha(\alpha + 1)$, can be rewritten by means of (2) and integration by parts to

$$
\alpha(\alpha + 1) \int_0^{2\pi} d\phi \int_{-1}^{1+1} d\xi P_\alpha(\xi) P_\alpha'[x\xi + \sqrt{(1 - x^2)} \sqrt{(1 - \xi^2)} \cos \phi] =
$$

$$
= - \int_0^{2\pi} d\phi \int_{-1}^{1+1} d\xi (1 - \xi^2) \frac{dP_\alpha(\xi)}{d\xi} P_\alpha(x\xi + \sqrt{(1 - x^2)} \sqrt{(1 - \xi^2)} \cos \phi)] +
$$

$$
+ \int_0^{2\pi} d\phi \int_{-1}^{1+1} d\xi (1 - \xi^2) \frac{dP_\alpha(\xi)}{d\xi} dP_\alpha'[x\xi + \sqrt{(1 - x^2)} \sqrt{(1 - \xi^2)} \cos \phi]
$$

where the following expressions were used for the unit vectors $n_1$, $n_2$ and $\mathbf{n}$

$$
n_1 = (0, 0, 1)
$$

$$
n_2 = (\sqrt{(1 - x^2)}, 0, x)
$$

$$
\mathbf{n} = (\sqrt{(1 - \xi^2)} \cos \phi, \sqrt{(1 - \xi^2)} \sin \phi, \xi).
$$

The function $P_\alpha(\xi)$ and its derivative in the point $\xi = 1$ is finite. On the other hand in the point $\xi = -1$ it holds [4] that

$$
\frac{dP_\alpha(\xi)}{d\xi} = \frac{\sin \pi \mathbf{n}}{\pi(1 + \xi)} \ (\xi \to -1).
$$

Therefore the first term on the righthand side of Eq. (3) reduces to

$$
4 \sin \pi \mathbf{n} . P_\alpha(-x).
$$

Simple rearrangement gives for the second term

$$
\int \left[ [n_1 \cdot n_2 - (n_1 \cdot \mathbf{n})(n_2 \cdot \mathbf{n})] P_\alpha'(n_1 \cdot \mathbf{n}) P_\alpha'(n_2 \cdot \mathbf{n}) \ d\Omega_n
$$

where the dashes at $P_\alpha$ denote derivation with respect to the corresponding variable.

We thus get

$$
\alpha(\alpha + 1) \int P_\alpha(n_1 \cdot \mathbf{n}) P_\alpha(n_2 \cdot \mathbf{n}) d\Omega_n = 4 \sin \pi \mathbf{n} . P_\alpha(-n_1 \cdot n_2) +
$$

$$
+ \int \left[ [n_1 \cdot n_2 - (n_1 \cdot \mathbf{n})(n_2 \cdot \mathbf{n})] P_\alpha'(n_1 \cdot \mathbf{n}) P_\alpha'(n_2 \cdot \mathbf{n}) \ d\Omega_n
$$

A similar equation is obtained by introducing the transformation $\mathbf{z} \to \mathbf{z}'$. By subtracting one equation from the other we get (1).