THE 4-PARTICLE WAVE FUNCTION IN THE SU\(_3\) SCHEME

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The building of 4-particle wave functions in the s—d nuclear shell, which are labelled by the partition \([22]\) of the unitary group \(SU_6\) (and of the permutation group), by the partitions \((42)\), \((04)\), \((20)\) of the \(SU_3\) group and by the angular momenta \(S = L = J = 0\), is shown. The Young operator and spherical tensor operator technique is used. The pairing in these wave functions is calculated.

I. INTRODUCTION

In paper [1] the \(SU_3\) classification was used to give the four-particle wave function of the \(O_{2\pi}\) nuclear ground state. Since the method applied was outlined very briefly and it is hoped it will be useful in establishing nuclear wave functions in general, a more detailed calculation is given in this paper.

The \(SU_3\) classification of the many particle states is induced by the special properties of the harmonic oscillator wave functions. It has been investigated from the point of view of the group theory in Elliot’s papers [2]. The \(U_6 \rightarrow SU_3 \rightarrow \mathbb{R}_3\) reduction for the s—d shell in the nuclei is shown in particular detail. Similar results have been obtained by Moshinsky [3] who uses the operator technique and the conception of canonical polynomials. In practical calculations it is possible to make use of both methods, which makes them easier.

In this connection the spherical tensor formalism becomes convenient since the abundance of the useful relations of the tensor calculus can be exploited. Thus we define creation and annihilation operators for oscillator quanta in spherical tensor form, the former being

\[
b_{l \pm 1} = \mp i(x \pm iy - ib^2p_x \pm b^2p_y), \quad b_{l0} = 2^{-\frac{1}{2}}(z - ib^2p_z),
\]

If we introduce the contravariant (and Hermitian conjugate) operators \(b^{1m}\) by

\[
b^{1m} = (-1)^{l+m} \sqrt{m} b_{1-m},
\]

the commutation relation becomes

\[
[ b^{1m_1}, b^{1m_2}_+ ] = 3b^2 \begin{pmatrix} 1 & 1 & 0 \\ m_1 & m_2 & 0 \end{pmatrix}.
\]

all the other non-trivial commutators being zero. Here the oscillator length unit

\[
b = \hbar \sqrt{m^{-\frac{1}{2}} \omega} = 41^{-\frac{1}{2}} A^{1/6} m^{-\frac{1}{2}} \hbar,
\]

where \(A\) is the atomic mass number, and the quantity \(\begin{pmatrix} : & \cdot & \cdot \end{pmatrix}\) is the well-known 3j-symbol. The ground state (vacuum) wave function is defined by \(b^{1m}|0\rangle = 0\), hence

\[
|0\rangle = \pi^{-\frac{1}{2}} b^{-\frac{3}{2}} \exp \left( -\frac{r^2}{2b^2} \right).
\]
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Then the single particle wave function for the 2s – 1d shell can be written as follows \((l = 0, 2)\)

\[
\psi_{im}^{(r)} = 2^{-\frac{1}{2}} b^{-2} \cdot (-1)^m (2l + 1)^\frac{1}{2} \cdot \sum_{m_1, m_2} \left( \begin{array}{cc} 1 & l \\ m_1 & m_2 - m \end{array} \right) b^+_{1m_1} b^+_{1m_2} |0\rangle.
\]

The group of unitary transformations \(U_6(SU_6)\) in the six-dimensional vector space of these functions can be represented in any many particle wave function space of the 2s – 1d shell. Since a classification of the many particle wave functions by irreducible representations of \(U_6\) leads at the same time to a classification by their permutation symmetry \([f]\), the many particle states can be labelled by the partition \([f]\) and in addition the wave functions by the corresponding Young tableau.

In a similar way the spherical tensors \(b^+_{1m}\) and \(\psi_{im}(r)\) induce the sub-groups \(U_3(SU_3)\) and \(R_3\) respectively of the group \(U_6\), so that the many particle states are still classified by their irreducible representations. The corresponding labels are the partition \((\lambda \mu)\) in Elliot’s notation [2] and the total angular momentum \(L\) respectively.

As is indicated, the orbital parts of the wave functions are investigated separately from their spin parts. However, these spin parts have the conjugate permutation symmetry \([f]\), if the resultant wave function is to be antisymmetric in the particle permutations.

The method by which the four-particle wave function is constructed is elucidated for the ground state function. It is assumed to have the smallest value of the total angular momentum and of the total particle spin and the maximum value of \(\lambda + \mu\). This means that \(J = S = L = 0\).

II. THE SPIN PART OF THE WAVE FUNCTION

This can easily be found using the results given in [4]. The only partition \([f]\) which belongs to the total four particle spin \(S = 0\) is \([f] = [2 2]\). The irreducible representation is two-dimensional, the basis functions being labelled by the standard Young tableaux \((2211)\) and \((2121)\), where the notation in Yamanouchi symbols is used. If they are built with Young operators, the particle spins are coupled two by two to the spin 1 and then they are coupled to the resultant spin \(S = [1 - 1] = 0\). Thus the \((2211)\)-basis function is given by

\[
\chi([2211); 1, 2, 3, 4) = \sum_{M=-1}^{1} \sum_{\sigma_1} (-1)^{1+M} \left(\begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & 1 \\ \sigma_1 & \sigma_2 - M & M \end{array}\right) \left(\begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & 1 \\ \sigma_3 & \sigma_4 - M & M \end{array}\right) \chi^{(1)} \chi^{(2)} \chi^{(3)} \chi^{(4)},
\]

where the single particle spin function is the usual one \(\chi_\sigma = \chi_{\frac{1}{2}\sigma}\) and, as below, the labelling of the zero spins or angular momenta is omitted. The second basis function \(\chi([2121); 1, 2, 3, 4)\) is obtained from the expression (6) by interchanging the spin variable \(\sigma_2\) and \(\sigma_3\) (or the particles 2 and 3).