ON THE ÚLEHLA-PETRÁŠ WAVE EQUATION

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The Petráš and the Úlehla wave equations of an anomalous electrically charged particle with spin one half are examined and their mutual relation as well as connections with the usual Dirac-Pauli equation are clarified.

INTRODUCTION

Petráš [1] and Úlehla [2, 3] proposed relativistic wave equations of the canonical form

\[(\gamma^\mu D_\mu - ia) \varphi = 0\]

which describe a particle with spin \(\frac{1}{2}\) but yet are not identical with the Dirac equation.

According to both authors the \(\gamma^\mu\) in (1) are twenty row square matrices and their elements depend on one free parameter. The methods which the two authors use when constructing their \(\gamma\)-matrices are, however, rather different and so are the explicit forms in which the matrices appear. Some elements of the Úlehla matrices are, indeed, singular for a certain value of the parameter whereas the elements of the Petráš matrices have no such singular point.

Both authors find that the Dirac-Pauli equation

\[(\gamma^\mu D_\mu - MF_\mu I^\mu(\gamma) - im) \varphi = 0\]

with \(I^\mu(\gamma) = \frac{1}{2} [\gamma^\nu, \gamma^\mu]\), \(F_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu\), and with \(M\) proportional to \(e/m\) is a consequence of their respective equations (1). Now, while Petráš [1] considers his equation (1) as simply equivalent to (2), Úlehla [3] concludes that there is no equivalence of his equation (1) with (2). Úlehla also finds that the possible value of the parameter in the case of his equation is limited to the intervals \((-\infty, 0)\) and \((0, \frac{1}{2})\). No such restriction occurs in the Petráš case.

1) Based on a diploma thesis submitted in partial fulfilment of the requirements for the degree of engineer-physicist.

2) We use natural units with \(\hbar = c = 1\), real space-time coordinates \(x^\mu = (x^0 = t, x^1, x^2, x^3)\) and metric tensor \(g_{\mu\nu}\) with \(g_{00} = 1, g_{11} = g_{22} = g_{33} = -1, g_{\mu\nu} = 0\) \((\mu \neq \nu)\). Further, as usual, \(D_\mu = \partial_\mu + ieA_\mu, \partial_\mu = \partial/\partial x^\mu\).

It will be seen below that neither Petrž's nor Śehla's investigations are sufficiently complete. Petrž neglected to show that his equation (1) can be derived from a Lagrangian. We amend it by giving explicit expressions for the metric matrix $A$ as well as the space-reflection operator $R$ and the operators of the infinitesimal Lorentz transformations $I^\nu$ corresponding to the Petrž form of (1). The Petrž equation (1) is truly equivalent to (2) for all (real) values of $M$, but only in case of $\varepsilon \neq 0, \kappa \neq 0$.

Šehla overlooked the possibility of also allowing values of the free parameter $> \frac{1}{2}$ in his equation (1). Again, if $\varepsilon \neq 0, \kappa \neq 0$, the Šehla equation (1) proves to be equivalent to (2) (and to the Petrž equation (1)) except for that single value of the ratio $Mm/\varepsilon = \frac{1}{2}$ for which the matrices $A^\nu$ in the Šehla original form are not defined. There exists a similarity transformation which transforms the Petrž representation of (1) into the Šehla representation for values of the ratio $Mm/\varepsilon$ except $\frac{1}{2}$. For this exceptional value the transformation becomes meaningless. This explains the origin and reveals the purely formal character of the singularity at $Mm/\varepsilon = \frac{1}{2}$ in the Šehla equation.

**THE PETRŽ EQUATION**

The matrices $A^\nu$ in the Petrž equation (1) can be written in the form

$$
A^\nu = \gamma^\nu \times q + a^\nu \times \beta^\nu.
$$

The meaning of the symbols in (2) is as follows: $\gamma^\nu$ are, of course, the Dirac matrices, $\beta^\nu$ are the Duffin-Kemmer $5 \times 5$-matrices, further

$$
q = -\beta^2 \gamma^2 \gamma^2 \gamma^2 = \frac{1}{2}(\beta^\nu \beta^\nu - 1),
$$

$$
a^\nu = a(\eta \sqrt{3}g^\nu \gamma + 2I^\nu(\gamma))
$$

where $\eta = \pm 1$ and $I(\gamma)$ = unit matrix of $\gamma$-algebra. The cross $\times$ denotes a direct product. The factor $a$ in (5) is a numerical parameter which must be either real or purely imaginary, i.e. $a^\nu = ea$, $e = \pm 1$.²)

Let us now introduce the wave functions

$\gamma^\nu$, $\beta^\nu$ fulfi the relations

$$
\gamma^\nu_{\mu} \gamma_{\nu} = 2g_{\mu}, \quad \gamma^\nu_{\mu} \beta_{\nu} = g_{\mu} \beta_{\nu} + g_{\nu} \beta_{\mu},
$$

$$
\beta^\nu_{\mu} \beta_{\nu} = 0 \quad (\mu \neq \nu, q \neq v).
$$

In conformity with our choice of the sign of $g_{\mu}$, the matrices $\gamma_{\nu}, \beta_{\nu}$ are Hermitian and $\gamma_{\nu}, \beta_{\nu} (k = 1, 2, 3)$ skew-Hermitian, so that

$$
\gamma^\nu_{\mu} = \gamma^\nu = \gamma_{\nu}^\mu \gamma_{\nu}, \quad \beta^\nu_{\mu} = \beta^\nu = (1 - 2g_{\nu}^2) \beta_{\mu}(1 - 2g_{\nu}^2).
$$

²) Petrž [1] uses two parameters $c_1, c_2$ which satisfy the condition $(c_1 + c_2)^2 = 3c_1^2$. They are connected with our parameter $a$ by the formulas

$$
c_1 = -ia(\eta \sqrt{3} - 1), \quad c_2 = -ia.
$$

The matrices $\gamma^\nu$ and $\beta^\nu$ fulfi the relations

$$
[\gamma^\nu, \gamma_{\nu}]_{\mu} = 2g_{\nu},
$$

$$
\gamma^\nu_{\mu} \beta_{\nu} = g_{\nu} \beta_{\nu} + g_{\nu} \beta_{\mu},
$$

$$
\beta^\nu_{\mu} \beta_{\nu} = 0 \quad (\mu \neq \nu, q \neq v).
$$

In conformity with our choice of the sign of $g_{\nu}$, the matrices $\gamma_{\nu}, \beta_{\nu}$ are Hermitian and $\gamma_{\nu}, \beta_{\nu} (k = 1, 2, 3)$ skew-Hermitian, so that

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$$