We present some results concerning the generalized homologies associated with nilpotent endomorphisms $d$ such that $d^N = 0$ for some integer $N \geq 2$. We then introduce the notion of graded $q$-differential algebra and describe some examples. In particular we construct the $q$-analog of the simplicial differential on forms, the $q$-analog of the Hochschild differential and the $q$-analog of the universal differential envelope of an associative unital algebra.

1 Introduction

Our aim is to discuss properties of nilpotent endomorphisms $d$ such that $d^N = 0$ for some integer $N \geq 2$, of the corresponding generalized homologies and of the associated generalizations of graded differential algebras. The natural setting would be to use a category of modules carrying representations of the group of $N$-th roots of the unity. Here, however, for simplicity, we shall work with complex vector spaces and the natural representations of this group in such spaces (i.e. multiplication by the corresponding complex numbers). Such a representation is characterized by a primitive $N$-root $q$ of the unity. For eventual applications to $q$-deformations, etc., we drop the assumption that $q$ is a root of unity and develop the notion of $q$-differential calculus for $q \in \mathbb{C}$ with $q \neq 0$ and $q \neq 1$ (i.e., $q \in \mathbb{C}\{0,1\}$). The terminology adopted here is influenced by a paper by M. Kapranov on similar topics [3]. Other examples of spaces with $d^N = 0$, etc. can be found in R. Kerner’s contribution to this Colloquium.

2 Generalized homology associated with $d^N = 0$

Let $E$ be a complex vector space equipped with a nilpotent endomorphism $d$ satisfying $d^N = 0$ where $N$ is an integer with $N \geq 2$. One has $\text{Im}(d^{N-k}) \subseteq \text{ker}(d^k)$ for $k \in \{0,1,\ldots,N\}$ and therefore the vector spaces $H^{(k)} = H^{(k)}(E) = \text{ker}(d^k)/\text{Im}(d^{N-k})$ are well defined. In fact $H^{(0)} = H^{(N)} = \{0\}$ and the $H^{(k)}$ are the generalization of the homology of $E$ for $1 \leq k \leq N - 1$.

Let $\ell$ and $m$ be two positive integers such that $\ell + m \leq N$. One has $\text{ker}(d^m) \subseteq \text{ker}(d^{\ell+m})$ and $\text{Im}(d^{N-m}) \subseteq \text{Im}(d^{N-(\ell+m)})$ so the inclusion $i^\ell : \text{ker}(d^m) \to \text{ker}(d^{\ell+m})$
induces a homomorphism \([i^t] : H^{(m)} \rightarrow H^{(\ell+m)}\). On the other hand, one has 
\[d^m(\ker(d^{\ell+m})) \subset \ker(d^{\ell})\] and 
\[d^m(\operatorname{Im}(d^{N-(\ell+m)})) \subset \operatorname{Im}(d^{N-\ell})\] and therefore \([d^m] : H^{(\ell+m)} \rightarrow H^{(\ell)}\). Notice that \([i^t] = [i] \ell\) and that 
\([d^m] = [d]^m\). One has the following lemma.

**Lemma 1** The hexagon \((H^{(m)}, \ell)\) of homomorphisms

is exact.

For the proof and for more details, we refer to [1] and [2]. We shall use the following

criterion ensuring the vanishing of the \(H^{(k)}\).

**Lemma 2** Let \(q\) be a primitive \(N\)-th root of the unity and assume that there is an
endomorphism \(h\) of \(E\) such that \(h \circ d - qd \circ h = I\) where \(I\) denotes the identity
mapping of \(E\) onto itself. Then one has \(H^{(k)} = 0\) for \(k \in \{1, \ldots, N - 1\}\).

In fact \(hd - qdh = I\) with \(q\) being a primitive \(N\)-th root of the unity implies, [2],
\[\sum_{k=0}^{N-1} d^{N-1-k} h^{N-1} d^k = [(N-1)!]_q I,\] where \([(N-1)!]_q = [(N-1)]_q \ldots [2]_q\) with
\([n]_q = 1 + q + \ldots + q^{n-1}\). The result follows since \([(N-1)!]_q \neq 0\).

**Example 1: Complex matrix algebras**

Let \(N\) be as above an integer with \(N \geq 2\) and let \(n_k, k \in \{1, \ldots, N\}\) be \(N\) integers
greater or equal to 1, \(n_k \geq 1\), with a sum \(S = \sum_{k=1}^{N} n_k \geq N\). The algebra of complex
\(S \times S\) matrices will be denoted by \(M_S(\mathbb{C})\). Associated to the family \((n_k)_i\), there is a
decomposition into rectangular blocks \(A^i_j\) of each element \(A\) of \(M_S(\mathbb{C}) : A = (A^i_j)\),
\((i, j = 1, 2, \ldots, N)\), where \(A^i_j\) is a complex matrix with \(n_i\) lines and \(n_j\) columns.
One equips \(M_S(\mathbb{C})\) with a \(\mathbb{Z}_N\)-graduation, \(M_S(\mathbb{C}) = \oplus_{p \in \mathbb{Z}_N} (M_S(\mathbb{C}))^p\), by giving
the degree \(j - i \mod(N)\) to the block \(A^i_j\), (i.e. to the matrix which has only \(A^i_j\)
as nonzero block). Equipped with this graduation, \(M_S(\mathbb{C})\) is a \(\mathbb{Z}_N\)-graded algebra:
\((M_S(\mathbb{C}))^{a}(M_S(\mathbb{C}))^{b} \subset (M_S(\mathbb{C}))^{a+b}, \forall a, b \in \mathbb{Z}_N\). Let \(e \in (M_S(\mathbb{C}))^1\) be such that \(e^N\)
is a multiple of the unit \(1 \in (M_S(\mathbb{C}))^0\):

\[e^N = \lambda 1, \ \lambda \in \mathbb{C}.\] \hfill (1)

Let \(q\) be a primitive \(N\)-th root of the unity and set \(d(A) = eA - q^a Ae\) for
\(A \in (M_S(\mathbb{C}))^{a}\). This defines an endomorphism \(d\) of \(M_S(\mathbb{C})\) which is of degree 1 and
satisfies

\[d(AB) = d(A)B + q^a Ad(B), \ \forall A \in (M_S(\mathbb{C}))^{a}, \ \forall B \in M_S(\mathbb{C})\] \hfill (2)