EXACT BOUND STATES FOR THE POTENTIAL
\[ V(r) = r^2 + \beta r^{-4} + \lambda r^{-6} \]
USING PARTIAL ALGEBRAIZATION TECHNIQUE

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Received 14 July 1994;
in revised form 10 April 1995

For the potential considered new sets of ground state and first few excited states are obtained. Also the defect of the wave function suggested by Kaushal and Parashar [Phys. Lett. A 170 (1992) 335] and Guardiola and Ros [J. Phys. A 25 (1992) 1351] has been shown.

Recently a lot of interest has been shown in the potential
\[ V(r) = ar^2 + \beta r^{-4} + \lambda r^{-6}, \quad a > 0, \quad \lambda > 0, \]
which is sometimes called the spiked anharmonic potential [1–7]. Kaushal and Parashar [6] have attempted to find exactly both the ground state and the excited states of this potential. However it is shown by Landtman [8] that their results are wrong. The exact ground states for the potential, however, can be found if the parameters \( a, \beta, \) and \( \lambda \) satisfy certain constraint. Very recently Varshni [9] has found four different possible sets of solutions for this potential using the wave function suggested by Kaushal and Parashar and also by Guardiola and Ros.

In this paper we find new sets of exact solutions for the ground state and the first or second excited state using partial algebraization method which has been applied to a class of problems called quasi exactly solvable problems [10–15] and for which only a finite number of bound states can be found.

We also show that the wave function suggested by Kaushal and Parashar is not the correct one for quasi exactly solvable cases. Instead of the algebraic expression \( 1 + \alpha_1 r^2 + \alpha_2 r^{-2} \) one should take \( \alpha_1 r^2 + \alpha_2 r^{-2} \), as the term independent of \( r \) here is exactly zero. If it is taken as non zero then \( \alpha_1 \) and \( \alpha_2 \) become infinite (Varshni finds them equal to order of \( 10^{15} \) and \( 10^{16} \), respectively).

Before we consider the potential mentioned above we describe here briefly the method of partial algebraization. Given a Schrödinger equation
\[ H \psi = E \psi \]
we perform an imaginary gauge transformation on the wave function $\psi(x)$ \cite{10}:

$$H \psi(x) \to \psi(x)e^{-f(x)}.$$ \quad (2)

Then,

$$H = -\frac{1}{2} \frac{d^2}{dx^2} + V(x),$$ \quad (3)

$$H_G = -\frac{1}{2} \frac{d^2}{dx^2} + A(x) \frac{d}{dx} + \Delta V,$$ \quad (4)

where

$$\Delta V = V(x) + \frac{1}{2} A'(x) - \frac{1}{2} A^2(x)$$ \quad (5)

while

$$f(x) = \int A(x') dx.'$$ \quad (6)

The gauge transformed eigenvalue equation reads

$$H_G \tilde{\psi}(x) = E \tilde{\psi}(x).$$ \quad (7)

Next we consider a finite dimensional representation of the SU(2) group with spins. The generators of the group are

$$T^+ = 2 j \xi - \xi^2 \frac{d}{d\xi}, \quad T^0 = -j + \xi \frac{d}{d\xi}, \quad T^- = \frac{d}{d\xi}.$$ \quad (8)

The corresponding finite dimensional representation is

$$R^j = (1, \xi, \xi^2, \ldots, \xi^{2j}).$$ \quad (9)

We choose the gauge in such a way that $H_G$ can be written as

$$H_G = \sum C_{ab} T^a T^b + \sum C_a T^a + \text{constant},$$ \quad (10)

where $a, b = (+, -, 0)$, and $C_{ab}, C_a$ are numerical coefficients. Using (8), Eq. 10 can be written as

$$H_G = -\frac{1}{2} P_4(\xi) \frac{d^2}{d\xi^2} + P_3(\xi) \frac{d}{d\xi} + P_2(\xi),$$ \quad (11)

where $P_n(\xi)$ denotes a polynomial of degree at most $n$ in $\xi$. To bring (11) in Schrödinger like form we put, say,

$$x = \int d\xi P_4^{-1/2}(\xi) = F(\xi),$$ \quad (12)

then

$$H_G = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{4} \frac{P_4^2 + P_3}{P_4^{1/2}} \frac{d}{dx} + P_2.$$ \quad (13)