REGULAR COLLIGATIONS FOR SEVERAL COMMUTING OPERATORS IN BANACH SPACE

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Several important results of the theory of operator colligations, stated initially for strict commutative colligations in Hilbert space [6] [7], are extended to the more general setting of regular commutative colligations in Banach space, which are defined in the text.

0. INTRODUCTION

The theory of operator colligations [1-5] has been further developed in the recent papers of M.S. Livšic [6], [7] in which colligations of several commuting operators in a Hilbert space and their applications to system theory are investigated. A close and far-reaching connection between this theory and the theory of algebraic curves was discovered. This connection finds its expression in the main results such as the generalized Cayley-Hamilton theorem for a system of several commuting operators in a Hilbert space and a method for constructing a canonical model of such a system.

In this paper we extend certain results of the theory to a wider class of commutative operator colligations. To facilitate the explanation, we first recall some terminology and basic facts related to the subject.

A colligation as defined by M.S. Livšic [2], [6] is a set of the form

\[ X = (A_1, \ldots, A_n; H, \varphi; E; \sigma_1, \ldots, \sigma_n) \]  \hspace{1cm} (1)

in which \( H \) and \( E \) are Hilbert spaces, \( A_k: H \to H, \varphi: H \to E, \sigma_k: E \to E \) (\( k = 1, n \)) are bounded linear transformations connected by the relations

\[ A_k - A_k^* = i\varphi^*\sigma_k\varphi, \quad k = 1, n. \]  \hspace{1cm} (2)
A colligation is called strict if \( \varphi \) maps \( H \) onto \( E \). It is called non-degenerate if \( \bigvee_k \ker \sigma_k = \{0\} \), or, what is the same, \( \bigvee_k \sigma_k E = E \). A strict non-degenerate colligation is in a certain sense a "minimal" colligation containing the given operators \( \{A_k\} \). In this case the space \( E \) is isomorphic to the subspace \( \bigvee_k (A_k - A_k^*) H = (\bigcap_k \ker (A_k - A_k^*))^\perp \) in \( H \).

A colligation is called commutative if \( A_k A_j = A_j A_k \); \( k, j = 1, n \).

To any strict commutative colligation \( X \) there correspond in a unique way two systems of linear operators \( Y_{kj}^{in} = \gamma_{kj}^{in}(X) \) and \( Y_{kj}^{out} = \gamma_{kj}^{out}(X) \); \( k, j = 1, n \) in \( E \) such that
\[
A_k A_j^* - A_j A_k^* = \gamma_{kj}^{in} \varphi K_{kj} \varphi,
\]
\[
A_j A_k^* - A_k A_j^* = \gamma_{kj}^{out} \varphi K_{kj} \varphi,
\]

\[
\gamma_{kj}^{out} = \gamma_{kj}^{in} + i(\sigma_k \varphi \sigma_j - \sigma_j \varphi \sigma_k)
\]
holds for any \( 1 \leq k, j \leq n \) [7]. Obviously,
\[
\gamma_{kj}^{in} = -\gamma_{jk}^{in}, \quad \gamma_{kj}^{out} = -\gamma_{jk}^{out}; \quad k, j = 1, n.
\]
(The superscripts "in" and "out" come from the system-theoretical interpretation of the theory, where the \( \gamma^{in} \) and \( \gamma^{out} \) are related in a natural way to the input and the output of the appropriate system.)

Let \( X \) be a strict commutative colligation with finite-dimensional space \( E \) and \( n = 2 \):
\[
X = (A_1, A_2; H, \varphi, E; \sigma_1, \sigma_2), \quad r = \dim E < \infty.
\]
Denote for simplicity \( \gamma^{in} = \gamma^{in}_{21}(X) \) and \( \gamma^{out} = \gamma^{out}_{21}(X) \). The discriminant function at the input of \( X \) is defined to be the polynomial
\[
D^{in}(x_1, x_2) = \det(x_2 \sigma_1 - x_1 \sigma_2 + \gamma^{in}).
\]
Similarly, the polynomial
\[
D^{out}(x_1, x_2) = \det(x_2 \sigma_1 - x_1 \sigma_2 + \gamma^{out})
\]
is called the discriminant function at the output of \( X \). Since \( \sigma_1, \sigma_2, \gamma \) are selfadjoint, the polynomials \( D^{in} \) and \( D^{out} \) have real coefficients.