THE CARATHÉODORY-FEJER EXTENSION THEOREM

Louis de Branges

A construction of Carathéodory and Fejér [1] produces a function which is bounded and analytic in the unit disk with specified initial coefficients. An operator generalization of the construction is now obtained for application to the invariant subspace problem. A formal proof [2] of the existence of invariant subspaces is given by the theory of square summable power series [3] in its vector formulation [4]. But the justification of the formal argument requires a determination of extreme points of a convex set [5]. A solution is now given to an extension problem for convex decompositions which arises in connection with the Carathéodory-Fejér theorem. A necessary condition for an extreme point is obtained as an application. The condition is conjectured to be sufficient.

A fixed Hilbert space \( \mathcal{C} \) of infinite dimension is used as a coefficient space. A vector is always an element of this space. An operator is a bounded linear transformation of vectors into vectors. If \( b \) is a vector, \( b^- \) is the linear functional on vectors defined by the inner product \( b^- a = \langle a, b \rangle \). If \( a \) and \( b \) are vectors, \( a b^- \) is the operator defined by \( (a b^-) c = a (b^- c) \) for every vector \( c \). A bar is also used for the adjoint of an operator. The absolute value symbol is used for the norm of a vector and for the operator norm of an operator.

The underlying Hilbert space is the space \( \mathcal{C}(z) \) of square summable power series \( f(z) = \sum a_n z^n \) with vector coefficients, \( \|f(z)\|^2 = \sum |a_n|^2 \). Other Hilbert spaces are constructed using power series \( B(z) \) with operator coefficients such that \( B(z) f(z) \) belongs to \( \mathcal{C}(z) \) whenever \( f(z) \) belongs to \( \mathcal{C}(z) \) and such that the inequality \( \|B(z) f(z)\| \leq \|f(z)\| \) is satisfied. An equivalent condition is that \( B(z) \) represents a function which is bounded by one in the unit disk.
If $B(z)$ is any such power series, define the $\mathcal{N}(B)$-norm of an element $f(z)$ of $\mathcal{C}(z)$ by
\[
\|f(z)\|_{\mathcal{N}(B)}^2 = \sup \left[ \|f(z) + B(z) g(z)\|^2 - \|g(z)\|^2 \right]
\]
where the least upper bound is taken over all elements $g(z)$ of $\mathcal{C}(z)$. The set $\mathcal{N}(B)$ of elements of $\mathcal{C}(z)$ of finite $\mathcal{N}(B)$-norm is a Hilbert space $\mathcal{N}(B)$ in the $\mathcal{N}(B)$-norm. The inclusion of $\mathcal{N}(B)$ in $\mathcal{C}(z)$ does not increase norms. If $f(z)$ is in $\mathcal{N}(B)$, then $[f(z) - f(0)]/z$ is in $\mathcal{N}(B)$ and the inequality
\[
\|f(z) - f(0)/z\|_{\mathcal{N}(B)}^2 \leq \|f(z)\|_{\mathcal{N}(B)}^2 - |f(0)|^2
\]
is satisfied. An important special class of spaces are those for which equality always holds in the inequality for difference-quotients.

Consider an arbitrary linear transformation $T$ of a Hilbert space $\mathcal{N}$ into itself, which is bounded by one, such that the identity $\|T^n f\|_{\mathcal{N}} = \|f\|_{\mathcal{N}}$ holds for no nonzero element $f$ of $\mathcal{N}$ for every nonnegative integer $n$. If the dimensions of the ranges of $1 - T^* T$ and $1 - T T^*$ do not exceed the dimension of the coefficient space $\mathcal{C}$, then $T$ is unitarily equivalent to the difference-quotient transformation $f(z)$ into $[f(z) - f(0)]/z$ in a space $\mathcal{N}(B)$ which satisfies the identity for difference-quotients.

A space $\mathcal{N}(B)$ need not be contained isometrically in $\mathcal{C}(z)$ even though the identity for difference-quotients is satisfied. A theory of minimal decompositions describes the interplay between $\mathcal{N}(B)$ and $\mathcal{C}(z)$ when $\mathcal{N}(B)$ is not contained isometrically in $\mathcal{C}(z)$.

If $f(z)$ is in $\mathcal{N}(B)$ and if $g(z)$ is in $\mathcal{C}(z)$, then $h(z) = f(z) + B(z) g(z)$ belongs to $\mathcal{C}(z)$ and the inequality
\[
\|h(z)\|^2 \leq \|f(z)\|_{\mathcal{N}(B)}^2 + \|g(z)\|^2
\]
is satisfied. Every element $h(z)$ of $\mathcal{C}(z)$ has a unique minimal decomposition for which equality holds. A given decomposition is minimal if, and only if, the identity
\[
\langle f(z), B(z) L(z) \rangle_{\mathcal{N}(B)} = \langle g(z), L(z) \rangle
\]
holds for every element $L(z)$ of $\mathcal{C}(z)$ such that $B(z) L(z)$ belongs to $\mathcal{N}(B)$. The identity for difference-quotients holds in a space $\mathcal{N}(B)$ if, and only if, the space contains no nonzero element of the form $B(z) c$ with $c$ in $\mathcal{C}$. 