THE CARATHÉODORY-FEJER EXTENSION THEOREM

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A construction of Carathéodory and Fejér [1] produces a function which is bounded and analytic in the unit disk with specified initial coefficients. An operator generalization of the construction is now obtained for application to the invariant subspace problem. A formal proof [2] of the existence of invariant subspaces is given by the theory of square summable power series [3] in its vector formulation [4]. But the justification of the formal argument requires a determination of extreme points of a convex set [5]. A solution is now given to an extension problem for convex decompositions which arises in connection with the Carathéodory-Fejér theorem. A necessary condition for an extreme point is obtained as an application. The condition is conjectured to be sufficient.

A fixed Hilbert space C of infinite dimension is used as a coefficient space. A vector is always an element of this space. An operator is a bounded linear transformation of vectors into vectors. If b is a vector, b^- is the linear functional on vectors defined by the inner product b^- a = <a,b>. If a and b are vectors, a b^- is the operator defined by (a b^-) c = a (b^- c) for every vector c. A bar is also used for the adjoint of an operator. The absolute value symbol is used for the norm of a vector and for the operator norm of an operator.

The underlying Hilbert space is the space C(z) of square summable power series f(z) = \( \sum a_n z^n \) with vector coefficients, \( \|f(z)\|^2 = \sum |a_n|^2 \). Other Hilbert spaces are constructed using power series B(z) with operator coefficients such that B(z) f(z) belongs to C(z) whenever f(z) belongs to C(z) and such that the inequality \( \|B(z) f(z)\| \leq \|f(z)\| \) is satisfied. An equivalent condition is that B(z) represents a function which is bounded by one in the unit disk.
If $B(z)$ is any such power series, define the $\mathcal{H}(B)$-norm of an element $f(z)$ of $\mathcal{C}(z)$ by

$$\|f(z)\|_{\mathcal{H}(B)}^2 = \text{sup} \left[ \|f(z) + B(z)g(z)\|^2 - \|g(z)\|^2 \right]$$

where the least upper bound is taken over all elements $g(z)$ of $\mathcal{C}(z)$. The set $\mathcal{H}(B)$ of elements of $\mathcal{C}(z)$ of finite $\mathcal{H}(B)$-norm is a Hilbert space $\mathcal{H}(B)$ in the $\mathcal{H}(B)$-norm. The inclusion of $\mathcal{H}(B)$ in $\mathcal{C}(z)$ does not increase norms. If $f(z)$ is in $\mathcal{H}(B)$, then $[f(z) - f(0)]/z$ is in $\mathcal{H}(B)$ and the inequality

$$\|\frac{f(z) - f(0)}{z}\|_{\mathcal{H}(B)}^2 \leq \|f(z)\|_{\mathcal{H}(B)}^2 - |f(0)|^2$$

is satisfied. An important special class of spaces are those for which equality always holds in the inequality for difference-quotients.

Consider an arbitrary linear transformation $T$ of a Hilbert space $\mathcal{H}$ into itself, which is bounded by one, such that the identity $\|T^n f\|_{\mathcal{H}} = \|f\|_{\mathcal{H}}$ holds for no nonzero element $f$ of $\mathcal{H}$ for every nonnegative integer $n$. If the dimensions of the ranges of $1 - T^* T$ and $1 - T T^*$ do not exceed the dimension of the coefficient space $\mathcal{C}$, then $T$ is unitarily equivalent to the difference-quotient transformation $f(z)$ into $[f(z) - f(0)]/z$ in a space $\mathcal{H}(B)$ which satisfies the identity for difference-quotients.

A space $\mathcal{H}(B)$ need not be contained isometrically in $\mathcal{C}(z)$ even though the identity for difference-quotients is satisfied. A theory of minimal decompositions describes the interplay between $\mathcal{H}(B)$ and $\mathcal{C}(z)$ when $\mathcal{H}(B)$ is not contained isometrically in $\mathcal{C}(z)$.

If $f(z)$ is in $\mathcal{H}(B)$ and if $g(z)$ is in $\mathcal{C}(z)$, then $h(z) = f(z) + B(z)g(z)$ belongs to $\mathcal{C}(z)$ and the inequality

$$\|h(z)\|^2 \leq \|f(z)\|_{\mathcal{H}(B)}^2 + \|g(z)\|^2$$

is satisfied. Every element $h(z)$ of $\mathcal{C}(z)$ has a unique minimal decomposition for which equality holds. A given decomposition is minimal if, and only if, the identity

$$\langle f(z), B(z)L(z) \rangle_{\mathcal{H}(B)} = \langle g(z), L(z) \rangle$$

holds for every element $L(z)$ of $\mathcal{C}(z)$ such that $B(z)L(z)$ belongs to $\mathcal{H}(B)$. The identity for difference-quotients holds in a space $\mathcal{H}(B)$ if, and only if, the space contains no nonzero element of the form $B(z)c$ with $c$ in $\mathcal{C}$. 