THEORETICAL POSSIBILITIES OF A SOLENOID AS MASS FILTER

P. Kubíček
Institute of Mining and Metallurgy, Ostrava*)

The paper deals with equations of motion of ions in a time-dependent magnetic field of a solenoid. These lead to a Hill equation approximated by a Mathieu equation. The instability of solutions of these equations in certain regions implies a theoretical possibility of filtering ion beams according to masses.

1. INTRODUCTION

We shall derive easily soluble equations of motion for ions in a uniform, generally time-dependent magnetic field of a sufficiently long solenoid. We start with the general equation of motion

\[
M \frac{dv}{dt} = Q(E + [v \times B])
\]

and the Maxwell equations

\[
\begin{align*}
\text{rot} \ E &= - \frac{\partial B}{\partial t} \\
\text{div} \ D &= 0 \quad D = \varepsilon_0 E
\end{align*}
\]

where \(M\) and \(Q\) are the mass and the charge of the ion, respectively. Magnetic field induced by the moving ions will be neglected.

Introducing cylindrical coordinates according to Fig. 1 with unit vectors \(r_o, \varphi_o, z_o\), and considering only magnetic field parallel to the \(z\)-axis, we find that equations (2) and (3) written in cylindrical coordinates are satisfied by the following components of the electric field \(E\):

\[
E_r = E_z = 0 \quad E_\varphi = - \frac{1}{2} \frac{\partial B}{\partial t} . r
\]

Considering further that \(dr_o = \varphi_o . d\varphi, d\varphi_o = -r_o . d\varphi, dz_o/dt = 0\), we obtain from the vector equation (1) a set of scalar equations for the components along \(z_o, \varphi_o\) and \(r_o\):

\[
\begin{align*}
\frac{dv_z}{dt} &= 0 \\
M \frac{dv_\varphi}{dt} + M v_r \frac{d\varphi}{dt} &= -Q \left( \frac{1}{2} \frac{\partial B_z}{\partial t} . r + v_r B_z \right) \\
M \frac{dv_r}{dt} - M v_\varphi \frac{d\varphi}{dt} &= Q . v_\varphi . B_z
\end{align*}
\]

*) Třída Osvoboditelů 31, Ostrava 1, Czechoslovakia.

Because \( v_r = r \cdot \frac{d\varphi}{dt} \) and \( v_r = \frac{dr}{dt} \), we get from (6) and (7)

\[
2M \frac{dr}{dt} \cdot \frac{d\varphi}{dt} + Mr \frac{d^2\varphi}{dt^2} = - \frac{1}{2} Q \left( \frac{\partial B_z}{\partial t} \cdot r + B_z \frac{dr}{dt} \right)
\]

\[
M \frac{d^2r}{dt^2} - Mr \left( \frac{d\varphi}{dt} \right)^2 = Q \cdot r \cdot B_z \frac{d\varphi}{dt}
\]

Integration of equation (8) (multiplied by an integrating factor \( r \)) leads to

\[
r^2 \frac{d\varphi}{dt} = - \frac{Q}{2M} B_z r^2 + \text{const}
\]

This equation must hold for any \( r \), therefore the constant must vanish. From equations (10) and (9) we obtain, denoting \( \frac{d\varphi}{dt} = -\omega_c/2 \)

\[
\omega_c = \frac{Q}{M} B_z(t)
\]

\[
\frac{d^2r}{dt^2} + \frac{\omega_c^2(t) \cdot r}{4} = 0
\]

Obviously \( \omega_c/2 \) is the angular velocity of the plane \( \sigma \) (see Fig. 1) in which the ion moves. This plane rotates about an axis defined by the magnetic line of force on which the ion was injected into the field. Equations (11) and (12) are basic equations for the study of the possibilities of mass separation in a solenoid.

2. THE FIELD OF A SOLENOID WITH SINUSOIDAL VARIATION IN TIME

Let the time-dependence of magnetic induction in the solenoid be given

\[
B_z(t) = B_{z0} + B_{z1} \cdot \cos \omega t
\]

Then from equation (12) with a substitution \( 2\xi = \omega t \) we obtain

\[
\frac{d^2r(\xi)}{d\xi^2} + \frac{e^2}{M^2\omega^2} \left( B_{z0}^2 + \frac{B_{z1}^2}{2} + 2B_{z0}B_{z1} \cos 2\xi + \frac{B_{z1}^2}{2} \cos 4\xi \right) \cdot r(\xi) = 0
\]

Let us introduce the symbols

\[
a = e^2(B_{z0}^2 + B_{z1}^2)/M^2\omega^2, \quad q = e^2B_{z0}B_{z1}/M^2\omega^2
\]

\[
\delta = e^2B_{z1}^2/2M^2\omega^2
\]

and substitute \( \xi = \pi/2 \) for \( \xi \); then equation (14) may be written as

\[
\ddot{r} + (a - 2q \cos 2\xi) \cdot r = -\delta r \cos 4\xi
\]