ANALYSIS OF RECTANGULAR PLATES LYING ON SOILS
WHOSE MODULUS OF DEFORMATION VARIES WITH DEPTH,
BY THE METHOD OF FINITE DIFFERENCES

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In [1], V. I. Solomin presented a solution using finite differences for a plate lying on a homogeneous half space. We examine below the case in which the modulus of deformation of the soil under the plate changes with depth according to the law

\[ E = E_0 + E_n z^n, \]  

(1)

in which \( E_0 \) is the initial modulus of deformation of the soil; \( E_n \) is the coefficient of increase of the modulus of deformation of the soil with depth; \( n \) is the nonhomogeneity index of the soil.

As regards the soil model, this article develops an idea by G. K. Klein [2]. For this variation of the modulus of deformation of the foundation bed, the settlement of its surface under the concentrated load \( P \) may be found from the equation

\[ \varpi = \frac{P}{\pi r (D_0 + D_n r^n)}, \]  

(2)

in which \( r \) is the distance from the point of application of the load to the point on the bed surface where the settlement is to be determined;

\[ D_0 = \frac{E_0}{1 - \mu^2}; \quad D_n = \frac{E_n}{(3 + n) \left( \frac{1}{1 + n} - \mu^2 \right)}; \]

\( \mu \) is the Poisson ratio of foundation soil, taken as a constant. The differential equation for the deflection surface of the plate on the elastic foundation bed is of the form

\[ \frac{\partial^4 W}{\partial x^4} + 2 \frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4} = \frac{p - q}{D}, \]  

(3)

in which \( W \) is the deflection of the middle surface of the plate; \( D = \frac{E_h^3}{12 (1 - \mu^2)} \) is the cylindrical rigidity of the plate; \( p \) is the intensity of external load; \( q \) is the intensity of soil reaction.

We will write Eq. (3) in terms of finite differences. For this purpose, we plot on the plate a regular grid with meshes \( \Delta x \) and \( \Delta y \) (Fig. 1). At the node \( k \), for \( \alpha = \Delta y^2 / \Delta x^2 \) we have

\[ ((3 \alpha^2 + 6 \alpha + 6) W_k - 4 (\alpha + 1) (W_a + \alpha W_b + W_c + \alpha W_d) \]  

\[ + 2 \alpha (W_e + W_f + W_g + W_h) + (W_i + \alpha^2 W_j + W_m + \alpha^2 W_l) \]  

\[ + \frac{q_k - q_b}{D} \Delta y^4 = 0. \]  

(4)
The reaction $q_k$ is expressed as displacements of the nodal points plotted on the plate. For this purpose, using Eq. (2) we find first the settlement of point $M$ on the surface, under a uniformly distributed load acting on a square with sides $c$ (Fig. 2) and having an intensity $P/c^2$.

This settlement is

$$W = \frac{P}{\pi D_0 c} \frac{F_z}{c}$$

$$F_z = \frac{1}{c} \int_{\infty}^{L} \int_{\infty}^{L} \frac{1 + \pi}{(x^2 + y^2)^{1/2}} + \frac{D_0}{D_0 (x^2 + y^2)^{1/2}}$$

The function $F_z$ may be calculated for any value of the nonhomogeneity index of the soil. Table 1 gives the values of this function for $n=1$. For $x/c=1.416$, $x/c=2.240$, and $x/c=2.832$, respectively, point $M$ has the following coordinates: $x_1=y_1=c$; $x_4=c$; $y_1=2c$; $x_1=y_1=2c$. In other cases, the coordinate $y_1$ is taken equal to zero, that is, point $M$ is taken on the $x_1$-axis. However, the tabular values of the function $F_z$ may be used also for points not located on the axis of the square, since the orientation of the area with respect to point $M$ for distances greater than $3c$ practically does not affect the values of this function.

The displacement of any node under all the reactions $X$ may be found from the following equation, written in matrix form

$$\vec{W} = \frac{1}{\pi D_0 c} \vec{A} \vec{X},$$

in which $\vec{W}$ and $\vec{X}$ are column vectors for the settlements and the reactions; $A$ is the coefficient matrix, and $c_\Delta = V \Delta x \Delta y$. For this case

$$\vec{W} = \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_S \end{bmatrix}, \quad \vec{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_S \end{bmatrix}, \quad \vec{A} = \begin{bmatrix} F_{11} & F_{12} & \cdots & F_{1S} \\ F_{21} & F_{22} & \cdots & F_{2S} \\ \vdots & \vdots & \ddots & \vdots \\ F_{S1} & F_{S2} & \cdots & F_{SS} \end{bmatrix}$$

The function $F_{\gamma \beta}$ is taken from Table 1, according to the distance between the nodes $\gamma$ and $\beta$ and the ratio $c_\Delta D_1/D_0$. When plotting the grid on the plate, it is convenient to use squares or almost square rectangles for the meshes.