CALCULATION OF BEAMS OF FINITE LENGTH ON AN INHOMOGENEOUS BASE

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The investigation of the interaction between structures and an inhomogeneous base is an urgent practical and rather complex theoretical problem. Therefore, approximate methods are used in articles devoted to a determination of forces in structures [1-6].

If we use the Winkler model of a base, the approximate solution of the problem stated can be obtained by means of the "small-parameter" method [5]. For this purpose we represent the change $C(x)$ of the natural inhomogeneity of the solid along the length of a beam as the sum

$$C(x) = C_0 + \varepsilon C^{(1)}(x),$$

(1)

where $C_0$ is the average value of the spring constant of the base; $\varepsilon C^{(1)}(x) = C_1(x)$ is a centered function describing the changes of inhomogeneity of the soil relative to $C_0$.

Writing the deflections of the beam $W(x)$ in the form of a series with terms $\varepsilon^n W^n(x)$, we substitute it and expression (1) into the known differential equation of bending of a beam. Separating in it terms with the same power with respect to $\varepsilon$, we obtain a chain of differential equations with constant coefficients for determining functions $\varepsilon^n W^n(x)$. All other quantities being sought are found from analogous series, and the boundary conditions for function $\varepsilon^n W^n(x)$ are found from the boundary conditions for function $W(x)$ being sought.

Assuming further, according to [5] et al., that the range of oscillations of $C_1(x)$ is small in comparison with $C_0$, we will restrict ourselves in the calculations to terms containing $\varepsilon$ in a power not greater than the first.

The zeroth approximation is the solution for homogeneous conditions, where for a load on the beam of $q_0 = \text{const}$ the settlement is determined from the relation

$$W_0 = \varepsilon q_0 / C_0,$$

(2)

and the forces are identically equal to zero.

For a quantitative description of the natural inhomogeneity of a base $C_1(x)$ over the length of the beam $l$ and its introduction into the calculation, it is most convenient to use an expansion of $C_1(x)$ in a Fourier series with a finite number of terms:

$$C_1(x) = \sum_{k=1}^{a} (A_k \cos 2k \pi x + B_k \sin 2k \pi x).$$

(3)

We note that the simultaneous use of the small-parameter method and Fourier expansion of $C(x)$ was carried out in [5, 6]. An investigation of a beam of finite length leads to specific difficulties.

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Denoting $\omega^2 = bC_0/EI$, where $b$ is the width of the beam, $EI$ is its flexural rigidity, we change to the dimensionless coordinate $\xi = x/l$ and write the equation of the elastic line of the beam for the first approximation $cW(1) (\xi) = W_1 (\xi)$, which with consideration of (3) will be

$$\frac{d^4 W_1 (\xi)}{d \xi^4} + \omega_0^4 l^4 W_1 (\xi) = \sum_{k=1}^{\infty} (a_k \cos 2k \pi \xi + \beta_k \sin 2k \pi \xi).$$

where $\omega_k = A_k/C_0; \beta_k = B_k/C_0$.

On the ends of the beam $\xi = \pm 1/2$ are the boundary conditions for the second and third derivatives of the solution of Eq. (4), which are written as

$$
\begin{align*}
M_1 (1/2) &= M_1 (-1/2) = -\frac{EI}{l^3} \left[ \frac{d^2 W_1 (1/2)}{d \xi^2} \right] \frac{1}{l^3} = 0; \\
Q_1 (1/2) &= Q_1 (-1/2) = -\frac{EI}{l^2} \left[ \frac{d^2 W_1 (1/2)}{d \xi^2} \right] \frac{1}{l^2} = 0.
\end{align*}
$$

Using the linear character of the equation of bending of a beam and the principle of superposition of the effect of various components following from it, we determine at first the settlements and forces for each term of sum (3) separately, and then, summing the results obtained, we obtain the total settlements, moments, and transverse forces.

The settlements $W_1^2 (\xi)$ and $W_1^3 (\xi)$, determined respectively by the consinosoidal and sinusoidal harmonic components of $C_1(x)$, are conveniently represented in the form of the products

$$W_1^2 (\xi) = H_2^2 (\xi) \, W_o \, a_k, \quad W_1^3 (\xi) = H_3^3 (\xi) \, W_o \, \beta_k.$$ 

For the $k$-th consinosoidal component of (3) we have

$$\frac{d^4 H_2^k (\xi)}{d \xi^4} + \omega_0^4 l^4 H_2^k (\xi) = \frac{a_k}{l^4} \cos 2k \pi \xi.$$ 

The general solution of Eq. (7) can be represented in the form

$$H_4^k (\xi) = R_1 \, e^{\lambda \xi} \cos \lambda \xi + R_2 \, e^{\lambda \xi} \sin \lambda \xi + R_3 \, e^{-\lambda \xi} \cos \lambda \xi + R_4 \, e^{-\lambda \xi} \sin \lambda \xi - \frac{1}{1 - 4 \nu_k^2} \cos 2k \pi \xi,$$

where $\nu_k = 2k \pi/\omega_0 l; \lambda = \omega_0 l/\sqrt{2}$.

Since the right side of Eq. (7) is even, the function $H_4^k (\xi)$ is also even, which is possible only if the equalities of the arbitrary constants $R_1 = R_3, R_2 = -R_4$ in the solution of Eq. (7) are valid.

We rewrite the expression for settlement (8) in the form

$$H_4^k (\xi) = R_1 \, \text{ch} \lambda \xi \cos \lambda \xi + R_3 \, \text{sh} \lambda \xi \sin \lambda \xi - \frac{1}{1 + \nu_k^2} \cos 2k \pi \xi.$$ 

and take its second and third derivatives:

$$\frac{d^2 H_4^k (\xi)}{d \xi^2} = \omega_0^2 l^2 \left[ R_2 \, \text{ch} \lambda \xi \cos \lambda \xi + R_4 \, \text{sh} \lambda \xi \sin \lambda \xi - \frac{\nu_k^2}{1 + \nu_k^2} \cos 2k \pi \xi \right];$$

$$\frac{d^3 H_4^k (\xi)}{d \xi^3} = \omega_0^3 l^3 \left[ \frac{1}{\sqrt{2}} (R_3 - R_1) \, \text{sh} \lambda \xi \cos \lambda \xi - \frac{1}{\sqrt{2}} (R_3 + R_1) \, \text{sh} \lambda \xi \sin \lambda \xi - \frac{\nu_k^2}{1 + \nu_k^2} \sin 2k \pi \xi \right].$$