HIGH-FREQUENCY STORAGE OF A BEAM
IN CYCLICAL ACCELERATORS *

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In the theory of high-frequency storage of a beam in accelerators with a constant magnetic field, it is important to take into account the disturbances of the already stored beam by successive cycles of acceleration. Such a disturbance leads, on the whole, to an increase in the energy spread of the stored particles and to a change of their mean energy. The general formulation of the problem and its solution for some special cases are presented in this article.

1. Equations of Motion

The change in energy of the particles is conveniently considered in the dimensionless variables

\[ z = \frac{2\pi Q}{eV} \int_{E_0}^{E} \frac{dE}{\Omega(E)}; \quad Q = \left| \frac{q\cdot\Omega_0 eV}{2\pi} \right|^{1/2}, \]

where \( \Omega(E) \) is the frequency of revolution of the particles of energy \( E \); \( \Omega' = \frac{d\Omega}{dE} \); \( q \) is the multiplicity; and \( V \) is the amplitude of the accelerating voltage. The subscript 0 refers to quantities corresponding to the storage energy. We denote by \( \varphi \) the phase of the accelerating field of frequency \( \omega(t) \) in which the particle is traveling:

\[ \varphi = q\Omega - \omega(t). \]

Later on it will be seen that this assumption holds for the great majority of particles. Furthermore, in the neighborhood of the storage energy, let the frequency of the accelerating field change in accordance with a linear law, increasing or decreasing with time.

Under the assumptions that have been made, the equations of motion have the simple form [1]

\[ \frac{dz}{dt} = \cos \varphi = -\frac{\partial H}{\partial q}; \quad H = -\sin \varphi + \frac{\Omega_0}{|\Omega_0|} z^2 - \tau z \cos \varphi; \]

\[ \frac{d\varphi}{dt} = \frac{\Omega_0}{|\Omega_0|} z - \tau \cos \varphi = \frac{\partial H}{\partial z}; \quad \cos \varphi = \frac{d\varphi}{dt} \left( \frac{2\pi}{q\Omega_0 eV} \right). \]

* This work was performed in 1958.
Here and in what follows we shall use as the independent variable the dimensionless time
\[ \tau = Q t. \] (1.5)

The beginning of the time scale is chosen in such a way that \( \omega(0) = \Omega_0. \)

It is seen from expressions (1.4) that the variables \( z \) and \( \varphi \) are canonically conjugate variables, which permit us at once to draw some general conclusions on the behavior of the beam on the basis of Liouville's theorem [1, 2]. In fact, as long as the frequency depends linearly on the time, there will exist on the phase cylinder \( (\varphi, \varphi) \) a region of closed phase trajectories (for \( |\cos \varphi_s| < 1 \)) traveling with a velocity equal to \( \cos \varphi_s. \) We denote the area of this stable region by \( S \) and assume that it is filled with accelerated particles of mean density \( \rho. \) At the end of each storage cycle, the frequency rapidly returns to its initial value, so that the particles remain close to the storage energy. According to Liouville's theorem, the mean density of the stored beam in phase space cannot be greater than \( \rho, \) and therefore we obtain for its energy width the estimate
\[ \delta_n \approx \frac{S_n}{2\pi}, \] (1.6)

where \( n \) is the number of cycles of storage. Inequality (1.6) goes over into an equality only if the entire phase space in the storage energy region is uniformly filled with particles of density \( \rho, \) or, as we shall express it, only for a "dense arrangement" of the separatrices. A deviation from the equality is considered in the following sections.

In what follows, we shall need the equation of the phase trajectories. From Eqs. (1.4) it follows that in the canonical variables \( y = \frac{\varphi}{\Omega_0} \) and \( \varphi \) we have
\[ y^2 - 2(\sin \varphi - \varphi \cos \varphi_s) = \text{const.} \] (1.7)

2. Change in the Energy of Particles in One Storage Cycle

After one cycle of storage, all particles experience displacements of the \( z \) coordinates, where the value of the displacement \( \Delta \) depends on the final coordinate \( z_f \) and on the phase \( \psi \) at which the particle should pass the separatrix (at \( y = 0 \)). At the point \( \psi \) the condition \( \frac{\varphi}{\Omega_0}(z) = \omega(\tau) \) is fulfilled, after which the sign of \( \dot{\varphi} \) reverses, since \( \dot{\varphi}(\psi) = 0. \) It is readily noted that the quantity \( \psi \) cannot take on any values in the interval \( 0-2\pi. \) This is demonstrated in the figure, where the rising separatrix and phase trajectories in the unstable region are shown schematically. The region of values which can be taken on by \( \psi \) is shown shaded, and is bounded on one side by the quantity \( 2\pi - \varphi_s, \) and on the other by the point \( \varphi_s \) determined from the equation
\[ \sin \varphi - \varphi \cos \varphi_s = -\sin \varphi_s + \varphi_s \cos \varphi_s. \] (2.1)

Using Eq. (1.7) we express the displacement of the coordinate \( \Delta \) as a function of the final phase \( \varphi_f \) and \( \psi; \)
\[ \Delta = \int_{\varphi_f}^\varphi \frac{2^{1/2} \cos \xi d\xi}{\left[\sin \xi - \sin \psi - (\xi - \psi) \cos \varphi_s\right]^{1/2}} \]
\[ + \begin{cases} 
2^{3/2} \int_{\varphi_f}^\varphi \frac{\cos \xi d\xi}{\left[\sin \xi - \sin \psi - (\xi - \psi) \cos \varphi_s\right]^{1/2}} \\
0, 
\end{cases} \] (2.2)

* Since for \( \tau = 0, \) the frequency can return to its initial value, not all particles pass through the phase \( \psi. \)