Enumerative Techniques for Solving Some Nonconvex Global Optimization Problems

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Summary. We give an overview on different methods for solving nonconvex minimization problems using techniques of enumeration of extreme points. The problems considered include concave, indefinite quadratic, and special structured problems such as the concave cost network flow problem and linear complementarity. For methods that enumerate the extreme points according to the ascending order of the value of a linear underestimating function we propose new techniques for obtaining such underestimators.

1. Introduction

The global minimization of a nonconvex function over a polyhedron is one of the harder problems in nonlinear optimization because of the potentially large number of local minima which may fail to be global. The general nonconvex global optimization problem has the form:

\[
\text{global min } f(x) \quad \text{subject to } x \in P
\]  

(1.1)

where \( f(x) \) is a continuous nonconvex function, and \( P \) is a bounded polyhedron in \( \mathbb{R}^n \). When \( f(x) \) is concave, problem (1.1) is referred to as a concave minimization problem. An important property of concave functions is that every local and global solution is achieved at some extreme point of the feasible domain. This property makes the problem more tractable since the search for global solutions can be restricted to the set of extreme points, even though this set in general may still be too large. Most of the problems considered here have a quadratic objective function. For general methods, applications, and references regarding nonconvex quadratic programming see the recent survey paper [28].

An obvious way to solve the concave programming problem in the case where the feasible domain is a polyhedral set, is complete enumeration of the extreme points. When the objective function is indefinite quadratic the solution may not be an extreme point. However vertex enumeration can be used to obtain approximate solutions. In the special case of a linear complementarity problem, if the problem is solvable, then it has an extreme point solution. Enumerating algorithms that exploit this special property can be used.

General techniques for total enumeration of the vertices of a linear polyhedron, are given by [5] (Chap. 18), Balinski [2], Burdet [6], Dyer and Proll [10], Manas and Nedoma [23], [7], [32], and Rossler [31]. The method of Manas and Nedoma is described in details with numerical examples in the book by Martos [24, Chap. 12]. A survey and comparison of methods for finding all vertices of a polyhedral set is given by Matheiss and Rubin [20] and Dyer [9]. For problems with large number of vertices total enumeration is computationally infeasible. It is important to mention here that, from the complexity point of view, the problem of reporting the number of vertices of a given polyhedron is \#P-complete [18], [36].

Enumeration of the extreme points of a linear polyhedron is closely related to the problem of ranking the extreme points in ascending or descending order of the
value of a linear or a linear fractional function [25], [33] defined on the polyhedron. While it is not impossible for a ranking method to degenerate to complete inspection of all vertices of the polyhedron, such a method can be used to provide approximation solutions, since it is quite often sufficient to obtain a near-optimal, rather than the exact, global optimum.

2. Global Concave Minimization by Ranking the Extreme Points

In this section we consider the concave minimization problem of the form:

\[
\text{global min } f(x), \quad x \in P
\]

(2.1)

where the objective function \( f(x) \) is concave and \( P \) is a bounded polyhedron in \( \mathbb{R}^n \).

**Theorem 2.1.** The solution to (2.1) problem occurs at some vertex of the polyhedron \( P \).

Since the global minimum of this problem occurs at some vertex of \( P \), one method for solving this problem is by searching among the extreme points of \( P \) using some linear underestimating function \( \Gamma(x) \), such that \( \Gamma(x) \leq f(x) \) for all \( x \in P \). In this approach we rank the extreme points in increasing order of the value of \( \Gamma(x) \) in order to obtain lower and upper bounds of the global minimum in such a way that the lower bounds keep increasing and the upper bounds keep decreasing during the execution of the algorithm.

Let \( \Gamma(x) \) be a linear underestimating function of \( f(x) \) over \( P \), that is

\[
\Gamma(x) \leq f(x), \quad x \in P
\]

To obtain lower and upper bounds for the global minimum of the (2.1) the following linear program is solved:

\[
\text{min } \Gamma(x) \quad \text{subject to } x \in P
\]

(2.2)

It is easily seen that if \( x_0 \) is an optimal solution of (2.2) and \( f^* \) is the global optimum of (2.1), then

\[
f_1 = \Gamma(x_0) \leq f^* \leq f(x_0) = f_u.
\]

The following simple proposition is fundamental for the solution procedure.

**Proposition 2.1.** Given any upper bound \( f_u \) of \( f^* \), denote by \( \{x_k\} \) to be the set of all extreme points \( x_k \) of (2.2) such that \( \Gamma(x_k) \leq f_u \). Then the original problem (2.1) has a global solution \( x^* \) such that \( x^* \in \{x_k\} \).

**Proof.** The global optimum of (2.1) problem occurs at some vertex of \( P \), and therefore is also a vertex of (2.2). Since \( \Gamma(x^*) \leq f_u \), the conclusion now follows.

We describe next a procedure [4], using the linear underestimating function \( \Gamma(x) \) and Murty's extreme point ranking approach:

1. Solve the linear program (2.2) to obtain \( x_0 \); take \( f_1 = \Gamma(x_0) \) as a lower bound of the global minimum \( f^* \).

2. Take \( f_u = f(x_0) \) as an upper bound on \( f^* \). Take \( x_0 \) as the "current best solution" to the original problem (2.1).

3. Use Murty's extreme point ranking procedure to find the next best extreme point solution \( x_k \) to (2.2). If \( \Gamma(x_k) > f_u \), then stop; the current best solution is the global solution to (2.2), and \( f^* = f_u \). If \( \Gamma(x_k) \leq f_u \), then replace \( f_1 \) by \( \Gamma(x_k) ; f_1 \) is a lower bound on \( f^* \).

4. If \( f(x_k) < f_u \), then replace \( f_u \) by \( f(x_k) \) and replace the current best solution to (2.1) by \( x_k ; f_u \) is an upper bound on \( f^* \). Otherwise return to step 3 without changing \( f_u \) or the current best solution.

A near optimum solution is obtained when the difference of the upper and lower bounds is small enough. The current best solution is a near optimum solution to the original problem. A similar approach has been used to obtain a global optimum of quasiconcave fractional problems [29], using ranking of the vertices in ascending order of the value of a linear fractional function.

In the above approach we considered Murty's extreme point ranking procedure. Murty's method is essentially based on the result that given the first, second, ..., \( k \)-th ranked extreme point of a linear programming problem, the next best \((k + 1)\)-th ranked extreme point is geometrically adjacent to at least one of these \( k \) points (see also [27]). Assuming that the problem is nondegenerate, the simplex algorithm may be used to maintain a list of such adjacent extreme points. In the presence of degeneracy however, one should also need to determine all basic representations of a degenerate vertex in order to use the simplex algorithm. Different methods have been proposed for handling the degenerate case (for an overview see [37]) and a variety of methods have been developed for ranking the extreme points of special structure polytopes (see for example [3]). Taha [35] uses also the idea of ranking the extreme points based on linear underestimating functions. The computational effectiveness of extreme point ranking algorithms is discussed in McKeown [21].