Duality of regularizations and hullfunctions for mathematical programming problems

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Abstract. For the study of mathematical programming problems and solution methods the duality theory forms a powerful tool. There are also some concepts of regularization and stabilization of a given problem for a better behavior in practical solution procedures. The aim of this paper is the investigation of duality aspects of such regularizations and the forming of hullfunctions on the other hand. Applications for handling of so-called ill-posed problems (Eremin) using some parametrizations of the original problem will emphasize the importance for practical numerical methods, especially. This results will inspire some applications to solution methods for parametric and multicriteria optimization.


Key words: Nonlinear programming, duality, solution methods, parametric programming, multicriteria optimization

1. Introduction

Starting point of the following investigations is the fact that solution methods for nonlinear programming problems will fail in general if the feasible set of the problem is empty or if the objective tends to infinity over the nonempty feasible set.

In such situations, however, we are needed any information for a correction of the original problem in “some minimal manner”. By introduction of parameters and additional functions we want to improve the properties of the given problem.

Well known notations from literature for this concept are stabilization, regularization, infimal convolution, hullfunction etc.

Our purpose is the study of properties of this concepts and the consequences for constructions of dual pairs of programming problems. So we will proof the duality of regularization and forming of hullfunction on the other hand.

Some applications, especially for the development of a general strategy for handling of so-called improper or ill-posed problems illustrate the importance for solution methods, too.

2. Notation and definitions

All notations in this paper are “classical definitions” from Convex Analysis (see Rockafellar [7]; Aubin [2]).

Let \( Y, Y^* \) be paired, locally convex, linear topological Hausdorff-spaces and if necessary partially ordered by some closed convex cones.

For any functional \( f : Y \rightarrow \mathbb{R} \), especially, we denote

- \( \text{epi} f := \{(y, \mu) \in Y \times \mathbb{R} \mid f(y) \leq \mu \} \) as epigraph of \( f \),
- \( \text{hypo} f := \{(y, \mu) \in Y \times \mathbb{R} \mid f(y) \geq \mu \} \) as hypograph of \( f \),
- \( f^* : Y^* \rightarrow \mathbb{R}, f^*(y^*) := \sup \{ \langle y, y^* \rangle - f(y) \} \) as convex-conjugate function of \( f \) in the sense of Fenchel and Rockafellar and
- \( f_\gamma : Y^* \rightarrow \mathbb{R}, f_\gamma(y^*) := \inf \{ \langle y, y^* \rangle - f(y) \} \) as concave-conjugate function of \( f \).

Obviously, it holds by definitions of the conjugate functions

\[
(-f)(y^*) = -f^*(-y^*), \quad (-f)^*(y^*) = -f(-y^*)
\] (2.1)
and
\[
[\tilde{f}]^*(y)^* = \lambda f^* \left( \frac{1}{\lambda} y^* \right) \forall \lambda > 0.
\]
\[\tag{2.2}\]

The concept of regularization of programming problems is in use for a long time. For example Arrow, Hurwicz and Uzawa in 1962 have got a better behavior for solution methods by addition of some quadratical terms into the original programming problem.

From geometrical point of view this is in some sense a "convexification" of the given problem functions by pointwise addition of convex or concave functions. (For more details of regularization and stabilization see also Vasil’ev [15].)

The \textit{infimal convolution} of two functions \(f, \psi: \mathbb{Y} \to \mathbb{R}\), analogous to the classical formula for integral convolution (see Rockafellar [7], Sects. 5, 16)
\[f \circ \psi \,(y) := \inf_{t \in \mathbb{Y}} \{f(t) + \psi(y - t)\}\]
forms in geometrical sense a "hull", instead, because this means an addition of the epigraphs of the functions:
\[f \circ \psi \,(y) := \inf_{t \in \mathbb{Y}} \{f(t) + \psi(y - t)\} = \inf \left\{ \mu \left( \begin{array}{c} \mu_1 \\ \mu_2 \end{array} \right) \in \text{epi } f + \text{epi } \psi \right\}. \quad \tag{2.3}\]

Now we want to generalize this concepts of regularization and hull, respectively, for using in duality theory of mathematical programming.

**Definition 2.1.** Let \(\psi: \mathbb{Y} \to \mathbb{R}\) be a proper convex, lower semicontinuous function with 
\[\psi(0) = 0, \quad \text{dom } \psi = \mathbb{Y}.\]

For functions \(f: \mathbb{Y} \to \mathbb{R}\) and \(h: \mathbb{Y}^* \to \mathbb{R}\) and any \(r > 0\) we denote
\[H_f, r: \mathbb{Y} \to \mathbb{R}, \quad H_f, r(y) := \inf_{t \in \mathbb{Y}} \{f(t) + r\psi(y - t)\}\]
as convex hullfunction of \(f\),
\[H_h, r: \mathbb{Y}^* \to \mathbb{R}, \quad H_h, r(y^*) := \sup_{t \in \mathbb{Y}} \{f(t) - r\psi(y^* - t)\}\]
as concave hullfunction of \(f\),
\[R_h, r: \mathbb{Y} \to \mathbb{R}, \quad R_h, r(y^*) := \psi(y^*) + \frac{1}{r} \psi^*(ry^*)\]
as convex regularization of \(h\),
\[R_h, r: \mathbb{Y}^* \to \mathbb{R}, \quad R_h, r(y^*) := \psi^*(y^*) + \frac{1}{r} \psi^*(-ry^*)\]
as concave regularization of \(h\).

In an analogous way to (2.3) we have for the concave hullfunction
\[H_f, r(x) = \sup \left\{ \mu \left( \begin{array}{c} \mu_1 \\ \mu_2 \end{array} \right) \in \text{hypo } f + \text{hypo } (-r\psi) \right\}. \]

Some properties are given in

**Lemma 2.1.** Let \(f, h\) and \(\psi\) be as in Definition 2.1, then one has
1. \(u_f, r = -H(-f), u_h, r = R_h, r\).
2. \(H_f, r(y^*) = h(y^*) - r^{-1} \psi^*(-ry^*)\).
3. \(\{f\text{ is proper and convex}\} \Leftrightarrow \{(H_f, r)^* = R(h, r)^* \} \).
   If \(f\) is in addition lower semicontinuous, then we have \(u_f, r = (R_f, r)^*\).
4. \(\{f\text{ is proper and concave}\} \Leftrightarrow \{(H_f, r)^* = R(h, r)^* \} \).
   If \(f\) is in addition upper semicontinuous, then we have \(u_f, r = (R_f, r)^*\).
5. \(H_f, r \leq f, \quad u_f, r \geq f \forall r \geq 0\).
6. \(\psi^*(y^*) \geq 0 \forall y^* \in \mathbb{Y}^*, \text{ then yields}\)
   \[R_h, r \geq h, \quad R_h, r \leq h \forall r > 0.\]

**Proof.** 1) is evident with \(\inf f = -\sup (-f)\).
2) follows with (2.1).
3) analogous to 4).
4) By definition is
\[H_f, r(y) = \inf \left\{ \langle y, y^* \rangle - \sup_{y_1, y_2 = y} \{f(y_1) - r\psi(y_2)\} \right\} = \inf \left\{ \langle y_1, y^* \rangle + \langle y_2, y^* \rangle - f(y_1) - r\psi(y_2) \right\} = h(y^*) + \inf \left\{ \langle y_2, y^* \rangle + r\psi(y_2) \right\} = f(y^*) - r\sup_{y \in \mathbb{Y}} \left\{ \langle y_2, -\frac{1}{r} y^* \rangle - \psi(y_2) \right\} = f(y^*) - r\psi^\prime\left(-\frac{1}{r} y^* \right) = R(h, r)^*(y^*) \text{ (with 2.1)}.
5) \((0, 0) \in \text{hypo}(\psi), \quad (0, 0) \in \text{hypo}(r\psi) \forall r \geq 0\).
6) is trivial with (2.1). \ \square

Because of the importance and the frequent use we define regularizations and hullfunctions also for the Euclidean norm, especially
\[\psi(y) := \frac{1}{2} \|y\|^2.\]

**Definition 2.2.** Let be \(\mathbb{Y} = \mathbb{R}^n\). For functions \(f: \mathbb{Y} \to \mathbb{R}\) and \(h: \mathbb{Y}^* \to \mathbb{R}\) and any \(r > 0\) we denote
\[V_f, r: \mathbb{Y} \to \mathbb{R}, \quad V_f, r(y) := \inf_{t \in \mathbb{Y}} \{f(t) + \|y - t\|^2\}\]
as convex Moreau-Yoshida-hullfunction of \(f\),
\[V_h, r: \mathbb{Y}^* \to \mathbb{R}, \quad V_h, r(y^*) := \sup_{t \in \mathbb{Y}} \{f(t) - \frac{1}{2r} \|y - t\|^2\}\]
as concave Moreau-Yoshida-hullfunction of \(f\),
\[T_h, r: \mathbb{Y} \to \mathbb{R}, \quad T_h, r(y^*) := \psi(y^*) + \frac{r}{2} \|y^*\|^2\]
as convex Tikhonov-regularization of \(h\),
\[T_h, r: \mathbb{Y}^* \to \mathbb{R}, \quad T_h, r(y^*) := \psi(y^*) - \frac{r}{2} \|y^*\|^2\]
as concave Tikhonov-regularization of \(h\).