ON ANALYTIC EQUIVALENCE OF OPERATOR POLYNOMIALS

Leiba Rodman

It is proved that analytic equivalence of monic operator polynomials implies the similarity of their linearizations. In particular, linear operator polynomials \( A - X \) and \( \lambda I - Y \) are analytically equivalent if and only if \( X \) and \( Y \) are similar.

Let \( B \) be a complex Banach space, and let \( \mathcal{L}(B) \) be the algebra of all (linear bounded) operators acting in \( B \). Polynomials of the form \( P(\lambda) = \sum_{r=0}^{\lambda} \lambda^j P_j \), \( \lambda \in \mathbb{C} \), \( P_j \in \mathcal{L}(B) \) will be called operator polynomials (o.p.).

The following problem has been proposed in [1]: Let \( X, Y \in \mathcal{L}(B) \), and let the following equality be satisfied,

\[
\lambda I - X = E(\lambda)(\lambda I - Y)F(\lambda), \quad \lambda \in \mathbb{C}
\]
where $E(\lambda)$ and $F(\lambda)$ are o.p. such that their inverses $(E(\lambda))^{-1}$ are also o.p. Is it true then that $X$ and $Y$ are similar (i.e. $X = S^{-1}YS$ for some invertible $S \in L(B)$)? For finite dimensional $B$ this question has a well-known affirmative answer.

In this note we provide the affirmative answer in a more general setting of analytic equivalence for monic o.p. To state the main result, we shall need the following definitions and notations.

An o.p. $P(\lambda) = \sum_{j=0}^{r} \lambda^j P_j$ is monic if $P_r = I$. For a monic o.p. $P(\lambda) = \lambda^r I + \sum_{j=0}^{r-1} \lambda^j P_j$ define its linearization $C_P \in L(B^n)$ as follows:

$$C_P = \begin{bmatrix}
0 & I & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & I \\
-P_0 & -P_1 & \cdots & -P_{r-1}
\end{bmatrix}.$$

The spectral properties of $P(\lambda)$ and $C_P$ are closely related (see [2, 4]). In particular, $\sigma(P) = \sigma(\lambda I - C_P)$, where $\sigma(P)$, as usual, denotes the spectrum of the o.p. $P(\lambda)$: $\sigma(P) = \{ \lambda \in \Phi \mid P(\lambda) \text{ is not invertible}\}$. O.p. $P(\lambda)$ and $Q(\lambda)$ will be called analytically equivalent relative to some open set $\Omega \subset \Phi$ if, for $\lambda \in \Omega$

$$P(\lambda) = E(\lambda)Q(\lambda)F(\lambda),$$