A NOTE ON MATRIX ELEMENTS OF FINITE \( U(n) \) TRANSFORMATIONS*)

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Matrix elements of finite transformations of the group \( U(n) \) in an arbitrary unitary irreducible representation of \( U(n) \) are discussed and a technique for their determination is presented. As a by-product a recursion relation for a special Racah coefficient of \( U(n) \) is obtained.

1. INTRODUCTION

For some time now, different groups \( U(n) \) have been widely used in Nuclear as well as High Energy Physics. They have been thought to carry the symmetry in some of their problems\(^1\). Since in this way different states of a physical system have been represented by the basis of unitary irreducible representation (UIR) of \( U(n) \) and the physical observables by the operators of \( U(n) \) it has been important to develop the Racah-Wigner calculus for \( U(n) \). In particular, to derive explicit forms of the Wigner coefficients and of the matrix elements of operators of \( U(n) \).

The matrix elements of the infinitesimal operators of \( U(n) \) in any UIR of this group have first been determined by Gel'fand and Zetlin \([2]\) by using a special canonical basis of UIR of \( U(n) \). This Gel'fand-Zetlin (G-Z) basis is a transcription of two facts. First, that according to Cartan any UIR of \( U(n) \) is uniquely determined by its highest weight, i.e. by the \( n \)-dimensional vector \( k_n = (k_{1n}, k_{2n}, \ldots, k_{nn}) \), where the \( k_{in} \) are integers satisfying the conditions

\[
(1.1) \quad k_{in} \geq k_{jn}, \quad i < j.
\]

And second, that any UIR of \( U(n) \) when restricted to \( U(n-1) \) is, in general, a reducible representation of \( U(n-1) \) which, according to Weyl, decomposes into a direct sum of UIR of \( U(n-1) \) characterized by \( k_{n-1} = (k_{1n-1}, k_{2n-1}, \ldots, k_{nn-1}) \) with the multiplicity 0 or 1. Thus repeating this procedure for the obtained

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\(^1\) E.g. in collective motions in nuclei, in calculations of the fractional parentage coefficients, in a motion of \( N \) particles in a common \( n \)-dimensional harmonic oscillator, in a construction of \( n \)-nucleon orbital states with permutational symmetry in the cluster model, in quark models of elementary particles, in Gell-Mann-Né'eman eightfold way, in \( SU(6) \) and \( SU(12) \) models of elementary particles, in a chiral symmetry approach etc. [1].
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UIR of $U(n - 1)$, and so on and taking into account that to any decomposition of a reducible representation into UIR corresponds a decomposition of the associated representation space we easily see that every orthonormal G-Z basis vector of the carrier space of UIR of $U(n)$ is characterized by the triangular pattern,

$$\langle k \rangle = \begin{vmatrix} k_{1n} & k_{2n} & \ldots & k_{n-1n} & k_{nn} \\ k_{1n-1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \cdots \\ k_{11} & \cdots & k_{1l} \\ k_{11} & \cdots & k_{1l} & \cdots \end{vmatrix}$$

where the integers $k_{ij}$, satisfy the conditions

$$k_{1l} \geq k_{2l} \geq \ldots \geq k_{ll}$$

and

$$k_{il} \geq k_{il-1} \geq k_{i+1l}, \quad i = 1, 2, \ldots, l - 1, \quad l = 1, 2, \ldots, n.$$

Gel'fand-Zetlin results on matrix elements of any infinitesimal operator of $U(n)$ in an arbitrary UIR published in 1950 [2] were later rederived by Baird and Biedenharn [3] and by Nagel and Moshinsky [4] by using algebraic infinitesimal methods and, in particular, the Schwinger-Bargmann boson-operator technique. Thus the matrix elements of the infinitesimal operators of $U(n)$ whose counterpart in the rotation groups $SO(3)$ are the matrix elements

$$\langle j^m | \ J_i | j^m \rangle,$$

of the angular momenta operators $J_i$, have been well known for a long time.

The situation with the matrix elements of finite operators of $U(n)$ — the analogous of Wigner $D$-functions for $SO(3)$ — is less satisfactory. These matrix-elements have been studied by Gel'fand and Graev [5] and in the particular case of $U(3)$, by Chacón and Moshinsky [6] in 1965\(^2\). In their approach, Gel'fand and Graev have studied the matrix elements of finite operators of the group $GL(n, \mathbb{C})$ rather than of $U(n)$, by using two different methods. In the first method, any transformation of $GL(n, \mathbb{C})$ is expressed as a product of transformations each belonging to a one-parameter subgroup; an explicit expression is then derived for the matrix elements of finite transformations for each one-parameter subgroup, i.e. they evaluate

$$\langle k' | e^{\lambda} | k \rangle,$$

\(^2\) The finite transformations of $SU(3)$ have been studied recently also by D. F. Holland [J. Math. Phys. 10 (1969), 531] using Nelson's parametrization and tensor calculus.