Ill-posed problems in structural optimization and their practical consequences

H. Baier*
Dornier GmbH, D-88038 Friedrichshafen, Germany

Abstract Structural optimization problems may be ill-posed or weakly conditioned. Different causes of such ill-conditioning are investigated, which can arise, for example, from the problem statement itself and its probable strong nonconvexity, or the interaction of the optimization algorithm with the discretized physical model. These effects could cause undesirable or incorrect solutions or no meaningful solutions at all. Thus ill-posed or ill-conditioned problems must be treated and assessed carefully. Some approaches to dealing with such problems and possible consequences for the optimal designs are discussed.

1 Introduction
Optimization of mechanical structures leads to nonlinear and nonconvex optimization problems, with usually implicit and relatively large system equations for the determination of response quantities as functions of optimization variables such as cross-sectional areas, thicknesses, shape, etc. The numerical solution of this kind of problem is usually computationally expensive. Moreover, these problems could be ill-posed or ill-conditioned, i.e. they may violate at least one of the Hadamard conditions, and/or may lead to undesirable consequences in the numerical treatment or the obtained results. Some of these weaknesses are investigated in this paper, with emphasis on their (numerical) treatment and especially on consequences of the (optimal) design itself. As will be shown, such weak conditions can be either man-made or problem inherent, and especially for the latter case, some interesting conclusions can be drawn. This discussion of ill-posed structural optimization problems has been triggered by a paper by Natke (1992) on ill-posed system identification problems.

2 Statement of the structural optimization problem
A typical but comprehensive problem statement is: minimize \( f = \text{weight}(x_1, x_2, \ldots, x_n) \), such that

\[
\begin{align*}
1 - u_i(x, t^*)/u_{i}^{a1} & \geq 0, \quad i \in I, \\
1 - \ddot{u}_j(x, t^*)/\dot{u}_{j}^{a1} & \geq 0, \quad j \in J, \\
1 - \sigma_k(x, t^*)/\sigma_{k}^{a1} & \geq 0, \quad k \in K, \\
\omega_{ql} & \leq \omega_q \leq \omega_{qu}, \quad q \in Q, \\
x_{ml} & \leq x_m \leq x_{mu}, \quad m = 1, \ldots, n, \\
s_p(u_i, \ddot{u}_j, \omega_{e}, \sigma_k, x) & = 0, \quad p = 1, \ldots, \\
\end{align*}
\]  

with time \( t^* \) such that \( \max r(t) = r(t^*) \), i.e. \( t^* \) provides the highest or most relevant response.

The design variable vector \( x^T = x_1, x_2, \ldots x_n \) must be determined to minimize weight such that displacements \( u_i \), accelerations \( \dddot{u}_j \) and stresses \( \sigma_k \) do not exceed given allowables (subscript \( a1 \)), while also eigenfrequencies and the design variables can be lower and upper bounded. For given design variables, the response data (state variables) are determined from the system equations \( s_p \), which are usually provided by the finite element algorithm (FEA). Since the response data can be a function of time, the constraints are assumed to be discretized in such a way that time steps \( t = t^* \) lead to the relevant maximum response. Various approaches for this are discussed by Hsieh and Arora (1986). Since the relevant time steps \( t^* \) usually change during the optimization steps, which also leads to some kind of ill-posedness, more stable time intervals \( \Delta t^* \) should be used instead, as outlined in Fig. 1.

![Fig. 1. Critical time intervals \( \Delta t^* \) ](image)

Problem (1) is non-linear in \( x \) and since the system equations \( s_p \) are usually numerically established by FEA it is also implicit. In order to save computational effort in the iterative solution process, the implicit FEA model is often substituted by intermediate explicit approximations, some consequences of which are discussed below within the context of ill-conditioning.

The nonconvexity of (1) could result in local optima. These are rare in the statically loaded truss-type structures often used in literature as benchmark examples for testing optimization algorithms, but occur more often in practical problems of fibre reinforced composite materials optimization or
when dynamic structural response is involved.

As far as ill-conditioning is concerned, it is assumed that at least a part regularization of (1) by proper scaling of optimization variables (e.g. all variables are of the order 1) and of the constraints functions \( g_j \) (e.g. \( 0 \leq g_j \leq 1 \)) is carried out. Moreover, if some functions, e.g. of type \( \sqrt{w(x)} \), are involved, for example, within a von Mises stress criterion, physical meaningless cases with \( w(x) < 0 \) also likely to cause software errors must be avoided by adding further side constraints of type \( w(x) \geq 0 \).

3 Hadamard conditions in structural optimization

Ill-posed problems are understood as those violating at least one of the following Hadamard conditions:

- the problem has a solution,
- the solution is unique,
- the solution changes only slightly with slight changes of problem parameters (e.g. bounds in constraints),
- or having other undesirable numerical or practical consequences. Most of these will be outlined in more detail in later sections in the context of real world applications, while in the following only a brief discussion on different reasons for violating these conditions will be given.

The existence of a solution is usually given for problems arising from practical tasks. However, in structural optimization, nonexistence could arise from contradictory constraints (empty feasible region) especially when constraints from quite different and technically complex sources (stress, dynamics, fabrication, ...) are to be formulated. After experiencing some unsuccessful iterations in that case to determine a feasible point, careful reexamination and redefinition of the problem is to be undertaken. In some instances, it is the too high ambiguity imposed on the structural performance (constraints) which generates such a contradiction, and this can be a valuable practical conclusion.

Another source of nonexistence of a solution is that the selected objective function has no finite lower bound within the feasible region. This could happen, for example, in the case of heat loaded structures with different materials, where changes in component stiffnesses with fixed stiffness relations do not change stresses. Thus also (infinitely) negative stiffnesses may provide "reasonable" stresses with infinitely low weight.

Some problems in generating a solution can also result from the interaction of the optimization algorithm with the discretized mechanical model, usually the FEA, for describing the system equations. The discretization could lead to erroneous or physical meaningless solutions (response and/or gradients) because of inadequate discretization, which could be even "amplified" by optimization; and unobtainable solutions due to degeneration of system equations (stiffness matrix) during the course of optimization steps.

These cases are discussed in more detail in Section 4.

Due to the nonconvexity of the structural optimization problem, local optima could arise and thus nonuniqueness has to be accepted to some extent. As will be shown, such local optima could frequently occur in the optimization of composite materials or in problems with considerations in dynamic response. Different approaches exist for trying to identify the global solution; these range from solving modified problems in order to "get rid" of an obtained local solution (Brarlin 1972), up to methods generating more or less randomly different starting vectors for the optimization algorithms. In practice, the use of different starting vectors selected also by engineering insight and taking then the best of all obtained solutions has proven so far to be the most reasonable approach. Quite often also flat optima occur, where feasible design variables significantly differing from the "mathematically" optimal ones lead to practically equivalent objective function values.

Continuity of the solution for continuous changes in problem parameters should be a basic proposition in optimal design or even be a basic design criterion. For example, slight deviations of the structural variables from their nominal values should also lead to slight and minimal deviations in relevant responses. Nevertheless, some kind of discontinuity could occur, for example, in topology or shape optimization, where slight changes in load or shape determine whether or not a component can be taken out from the design. Such effects are then even more relevant when gradients are involved such as in the use of the Kuhn–Tucker–conditions, which is discussed in Section 7.

4 Ill-posed system equations during optimization steps

The system equations \( s_p \) in (1) describe the physical behaviour of the structure and establish – usually implicitly and numerically – the relation between optimization variables and response quantities. These equations could degenerate and become ill-conditioned during optimization steps, even for well-behaved starting designs. Such potential sources of ill-conditioning are degenerated stiffness matrices, e.g. due to design variables approaching critical lower values (cross-sectional areas or thicknesses close to zero), or due to highly distorted finite elements generated, e.g. in shape optimization. To overcome this problem, proper accuracy measures to avoid degeneration are to be checked with eventual update or even complete re-establishment of the finite element mesh. A simple, albeit not necessarily perfect, criterion for geometric design variables such as plate thicknesses \( t_i \) is, for example,

\[
t_i \geq \epsilon \max_j t_j
\]

with \( \epsilon \approx 10^{-4} \), i.e. the smallest thickness should be at least \( 10^{-4} \) of the largest. By this, an ill-conditioned stiffness matrix could be avoided, if the stiffness matrix condition number itself is not constrained.

In order to avoid element degeneration, mesh control must be carried out to some extent during shape optimization. Although this is less critical in the case when "natural" shape modes are used as the base for shape variation with the mode amplitudes as optimization variables, such as outlined by Zhang and Belegundu (1992), mesh control in the end is also required there. A criterion as proposed by Zhang and Belegundu (1992) is

\[
4 \det J/A > 0 ,
\]

\[
8 \det J/N > 0 .
\]