FE-shape sensitivity of elastoplastic response

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Abstract A new approach for performing FE-shape design sensitivity analyses (DSA) of structural models with both linear elastic and elastoplastic material behaviour is presented. In the formulation of the method the derivation of the FE equilibrium equations is performed analytically leading to various terms. The differentiation of some parts of these terms is determined numerically, therefore the method is semianalytical. The formulation is particularized for isoparametric finite elements for which "exact" numerical differentiation can be obtained (exact up to round-off error). Examples of some plane stress problems testify that the results of the new method are not dependent either on the size of perturbation of the design variables or on the FE mesh refinement among other factors.

1 Introduction

The importance of FE-shape DSA has attracted much interest recently as demonstrated by the constant flow of literature.

The most commonly used technique for obtaining shape DSA for FE structural models with linear elastic, static response is based on implicit differentiation of the related equilibrium equations with respect to the design variables. In the traditional approach this implicit differentiation indicates that the determination of the nodal displacement sensitivities requires that the design sensitivity of the global stiffness matrix is known as shown by Zienkiewicz and Campbell (1973). The methods for determining the latter sensitivities are analytical or semianalytical depending on whether they are determined analytically, before their numerical evaluation, or by numerical differentiation. Substantial numerical errors have been reported by Barthelemy and Haftka (1990), Pedersen et al. (1990) and Olhoff and Rasmussen (1991) when the semianalytical method is applied to structures where the displacement field is predominantly a rigid-body rotation relative to the actual deformation of the finite element.

Olhoff et al. (1993) present an "exact" method for performing numerical differentiation of a vast range of stiffness matrices. The application of this method eliminates problems of severe dependence of semianalytical DSA on the size of perturbation of the design variables and on the FE mesh refinement.

Shape DSA for FE structural models with elastoplastic response seems to be far more complicated than for linear elastic response. Tsay and Arora (1990) give a general theory of DSA for nonlinear structures and Tsay et al. (1990) reduce it to several special analytical cases. Kleiber et al. (1995) and Kleiber and Kowalczyk (1995) present the general problem of sizing, material and loading parameter sensitivity for nonlinear structures and examples are shown.

In this paper a general method for shape DSA for FE-structural models with both linear elastic and elastoplastic material behaviour is presented. The starting point of the present formulation is the generalized FE-equilibrium equation, which is valid for both the linear and nonlinear structural FE models. This approach does not require the determination of the sensitivities of the global stiffness matrix with respect to the design variables as is the case in the traditional methods as presented by Zienkiewicz and Campbell (1973). The "exact" method for numerical differentiation presented Olhoff et al. (1993) is incorporated in the current approach. The above-mentioned benefits (i.e. the use of this technique introduced for the DSA of problems with linear elastic material behaviour) are maintained in the proposed method when DSA is extended to structural models with elastoplastic response.

In Section 2 the basic features of the traditional method of "exact" semianalytical shape DSA for FE structural models with linear elastic material behaviour presented by Olhoff et al. (1993) are reviewed.

In Section 3 the procedure used in this work for the incremental elastoplastic FE-analysis is presented.

In Section 4 the formulation for the new generalized "exact" semianalytical method for shape DSA is shown. The term generalized is used in order to stress that the method may be applied for structural models with both linear elastic and elastoplastic material behaviour.

Finally, examples of structures represented by plane stress models with linear elastic perfectly plastic von Mises material behaviour are presented and discussed in Section 5. The applications can, however, be easily extended to other types of structures and material nonlinearities.

Section 6 summarizes the main conclusions of the present work.

2 The traditional "exact" semianalytical shape DSA

The primary goal of DSA is to determine the sensitivities $\partial u / \partial s_i$ of nodal displacements $u$ of the FE model with respect to the shape design variable $s_i$ of the geometric model.
The shape design variables $s_i$ define the geometric model of the structural system while the nodal coordinates $a_j$ define the FE model for the structural analysis. The FE model however is defined for the given geometric model.

Therefore the chain rule must be applied for the derivation of the equilibrium equations of the FE model with respect to $s_i$. At first the equations must be derived with respect to $a_j$, the nodal coordinates of the FE structural model, and then the result must be multiplied by $\partial a_j/\partial s_i$. The calculation of this value is usually obtained from the geometric model by means of finite difference techniques as discussed in detail by Sienz (1994). This will not be discussed further in this paper.

The derivation with respect to $a_j$ is obtained as follows. One starts from the FE equilibrium equation

$$\mathbf{K} \mathbf{u} = \mathbf{f}, \quad (1)$$

where $\mathbf{u}$ is the stiffness matrix, $\mathbf{K}$ is the nodal displacement vector and $\mathbf{f}$ is the vector of the nodal forces.

To reach our goal an implicit differentiation of (1) is performed,

$$\frac{\partial \mathbf{u}}{\partial a_j} = \mathbf{K}^{-1} \mathbf{p}_j, \quad (2)$$

where

$$\mathbf{p}_j = \frac{\partial \mathbf{K}}{\partial a_j} \mathbf{u} + \frac{\partial \mathbf{f}}{\partial a_j}, \quad (3)$$

is the vector of pseudo-forces. In general $\mathbf{f}$ is independent of $a_j$, in which case the last term in (3) may be considered equal to zero.

The determination of $\partial \mathbf{u}/\partial a_j$ requires that $\partial \mathbf{K}/\partial a_j$ be known. If the terms of $\partial \mathbf{K}/\partial a_j$ are determined by numerical differentiation, the term semianalytical sensitivity analysis is applied.

Many authors, such as Barthelemy and Haftka (1990), Pedersen et al. (1990) and Olhoff and Rasmussen (1991), have reported serious inaccuracies in some problems of semianalytical sensitivity analysis. Olhoff et al. (1993) present a simple procedure for the elimination of error in this kind of analysis for various finite elements, in particular, isoparametric elements. These elements admit an "exact" numerical differentiation of the stiffness matrix by means of computationally inexpensive first-order finite differences.

The main features of this procedure are reviewed in the following.

The isoparametric element stiffness matrix $\mathbf{K}_e$ in $x - y$-global coordinates is given by

$$\mathbf{K}_e = \int_{\Omega} \mathbf{B}^t \mathbf{D} \mathbf{B} |\mathbf{J}| \, d\Omega, \quad (4)$$

where $\Omega$ is the domain of the element described in curvilinear, nondimensional $\xi - \eta$ coordinates for the element, $|\mathbf{J}|$ is the determinant of the Jacobian matrix, $\mathbf{B}$ is the strain-displacement matrix and $\mathbf{D}$ the elasticity matrix. The general form of (3) is nonlinear, so direct application of a finite difference approach cannot yield "exact" results and is a cause of inaccuracy. Differentiation of (4) with respect to the design variable $a_j$ yields

$$\frac{\partial \mathbf{K}_e}{\partial a_j} = \int_{\Omega} \mathbf{B}^t \mathbf{D} \mathbf{B} |\mathbf{J}| \, d\Omega + \int_{\Omega} \mathbf{B}^t \frac{\partial \mathbf{D}}{\partial a_j} \mathbf{B} |\mathbf{J}| \, d\Omega +$$

$$\int_{\Omega} \mathbf{B}^t \frac{\partial \mathbf{B}}{\partial a_j} |\mathbf{J}| \, d\Omega. \quad (5)$$

Since $\mathbf{D}$ is not a function of $a_j$, the second term on the right-hand side can be dropped. Considering that $\mathbf{D}$ is symmetric, (5) can be expressed after some manipulation as

$$\frac{\partial \mathbf{K}_e}{\partial a_j} = 2 \int_{\Omega} \mathbf{B}^t \frac{\partial \mathbf{B}}{\partial a_j} |\mathbf{J}| \, d\Omega, \quad (6)$$

where $\mathbf{B}$ is defined as

$$\mathbf{B} = \frac{\partial \mathbf{B}}{\partial a_j} + \frac{\mathbf{D}}{2|\mathbf{J}| \partial a_j}. \quad (7)$$

The "exact" numerical differentiation in (5) depends on the "exact" numerical differentiation of $\mathbf{B}$ and $|\mathbf{J}|$ with respect to $a_j$. As was pointed out by Olhoff et al. (1993), since the determinant of $\mathbf{J}$ is either independent or a linear function of any nodal coordinate $a_j$, its "exact" derivative with respect to $a_j$ can be computed numerically by means of

$$\frac{\partial |\mathbf{J}|}{\partial a_j} = \frac{\Delta |\mathbf{J}|}{\Delta a_j} = \frac{|\mathbf{J}[(1 + \eta_j)a_j]| - |\mathbf{J}(a_j)|}{\eta_j a_j}, \quad (8)$$

where $\eta_j$ is the perturbation parameter associated with $a_j$ and

$$\eta_j = \frac{\Delta a_j}{a_j} > 0. \quad (9)$$

The computation of $\partial \mathbf{B}/\partial a_j$ calls for the differentiation of the derivatives of the interpolation functions $N_{i,x}$ and $N_{i,y}$ as defined below,

$$\{ N_{i,x} \} = \Gamma \{ N_{i,x} \}, \quad i = 1, \ldots, n, \quad (10)$$

where the matrix $\Gamma = \mathbf{J}^{-1}$, (11) is the inverse of the Jacobian matrix. Components of $\Gamma$ are complicated functions of $a_j$ and cannot be differentiated exactly via numerical procedure. To circumvent this problem the identity matrix

$$\mathbf{I} = \Gamma \mathbf{J}, \quad (12)$$

is differentiated yielding

$$\frac{\partial \Gamma}{\partial a_j} = -\Gamma \frac{\partial \mathbf{J}}{\partial a_j}. \quad (13)$$

The components of $\mathbf{J}$ are either independent or a linear function of any design variable variable $a_j$ so $\partial \mathbf{J}/\partial a_j$ can be determined "exactly" via numerical differentiation

$$\frac{\partial \mathbf{J}}{\partial a_j} = \frac{\Delta \mathbf{J}}{\Delta a_j} = \frac{|\mathbf{J}[(1 + \eta_j)a_j]| - |\mathbf{J}(a_j)|}{\eta_j a_j}, \quad (14)$$

The computation of (5) can then be performed using "exact" numerical differentiation.

### 3 Incremental elastoplastic FE-analysis

In this section the procedure used in this work for the incremental elastoplastic FE-analysis is examined. At the constitutive level the stresses in the plastic regions are obtained from the solution of a mathematical programming problem. At the structural level the FE equilibrium equations are solved by means of a quasi-Newton BFGS-method with displacements as unknowns. This procedure was successfully used by Eboli (1994) for plate bending problems and was adapted here for plane stress problems.