Technical Papers

Simultaneous structural analysis and design based on augmented Lagrangian duality

T. Larsson and M. Rönqvist
Department of Mathematics, Linköping Institute of Technology, S-581 83 Linköping, Sweden

Abstract Solution procedures in structural optimization are commonly based on a nested approach where approximations of the analysis and design problems are solved alternately in an iterative scheme. In this paper, we study a simultaneous approach based on an integrated formulation of the analysis and design problems. An advantage of the simultaneous approach, when compared to the nested one, is that the dependence between the analysis and design variables is imposed explicitly. In the nested approach, this dependence is implicitly determined through the solution of the analysis problem. Earlier simultaneous approaches mostly utilize various penalty function reformulations. In this paper, we make use of two augmented Lagrangian schemes, which avoid the numerical ill-conditioning inherent in penalty reformulations. These schemes give rise to Lagrangian subproblems with somewhat different properties, and two efficient techniques are adapted for their solution. The first is a projected Newton method, and the second is a simplicial decomposition scheme. Computational results for bar-truss structures show that the proposed schemes are viable approaches for solving the integrated formulation, and that they are promising for future developments.

1 Introduction

Structural optimization, or optimal design, deals with the problem of designing a mechanical structure in an efficient way with respect to some criterion, while taking design restrictions into account. A mathematical model of a structural optimization problem consists of two connected optimization problems: the design problem that determines sizes, materials or shapes, and the analysis problem that describes the response of the structure when affected by loads by finding its minimal potential energy. The overall problem can be formulated as a two level problem, where the analysis problem is on the lower level. This two level mathematical problem is, in general, prohibitively difficult to solve directly.

The most common solution scheme is to use an iterative nested approach where the analysis and the design problems are approximated and solved alternately. Given a design, the approximate analysis problem is, in general, formulated and solved by the Finite Element Method (FEM). Then the analysis problem is equivalent to finding a solution to a system of equations; this system is known as the state, or equilibrium, equations. This system becomes linear or non-linear depending on whether or not linear elasticity may be assumed. The next step is to formulate an approximate design problem in order to find an improved design. This is accomplished by approximating the constraint functions with respect to the design variables by means of Taylor series expansions. However, the sensitivity analysis, i.e. the computation of derivatives, is generally computationally expensive due to their implicit dependence on the analysis variables via the state equations. The approximate problem is then solved by an adequate optimization method (see e.g. Belegundu and Arora 1985a, b). The nested approach is suitable for connecting the existing well-developed finite element codes with an optimal design process. Furthermore, the general availability of such analysis systems is one of the main reasons for the popularity of this approach.

An alternative to the nested approach is to state the analysis and design problems in an integrated formulation and solve them simultaneously; this is the so-called simultaneous approach. A major difference between this and the nested approach is that the implicit dependence between the analysis and design variables is here instead imposed explicitly. The simultaneous approach has an advantage if we consider non-linear elasticity since the analysis problem is solved successively as the design is altered; hence the non-linearities are imposed smoothly during the solution process. One difficulty inherent in simultaneous schemes is that a larger problem, involving both analysis and design variables, has to be solved. The simultaneous approach has so far not been extensively used, although some attempts have been made.

The idea of integrating structural analysis and design was first introduced by Fox and Schmit (1965, 1966). They studied the case of linear elasticity and used a Conjugate Gradient (CG) minimization technique for solving the integrated problem reformulated by an exterior penalty function technique. They concluded that the numerical ill-conditioning of the penalty formulation caused a rather poor convergence performance. Fuchs (1982, 1983) studied bar-truss structures and formulated an explicit integrated formulation, which was solved by an interior penalty method. For solving the sequence of unconstrained problems he used various CG techniques. Haftka (1985) suggested an interior penalty reformulation and applied a preconditioned CG method. The use of preconditioning techniques reduced the difficulties caused by ill-conditioning. Both linear and non-linear analyses were considered.

Shin et al. (1988) considered the simultaneous approach to solve eigenvalue problems. Here, a Newton method with a finite-difference approximation of the Jacobian was used. SMAUTI and Schmit (1988) studied geometrically non-linear truss structures, and used the Generalized Reduced Gradient (GRG) method to solve an integrated formulation. Haftka and Kamat (1989) treated non-linear elastic bar-truss struc-
tures and proposed two solution procedures. First, a CG procedure is used for solving an extended interior penalty reformulation. Second, the Sequential Quadratic Programming (SQP) method is applied to the integrated formulation. Barthelmey et al. (1991) studied the problem of defining a hole in a plate. This application involves relatively few design variables. A preconditioned CG method is used for the solution. Ringertz (1992) considered non-linear shell structures, and solved an interior penalty reformulation with a Newton method. The use of an SQP method for problems including non-linear analysis has also been studied by Orozco and Ghattas (1992). They concluded that sparse implementations of an SQP method constitute efficient solution procedures for the integrated formulation.

Techniques related to the simultaneous approach have been proposed by Wu and Arora (1987), and Hafkka (1989). They studied applications with non-linear analysis and suggested solution procedures where the analysis and design processes are, in a sense, integrated. The analysis is performed in a number of increments, and in each of these an optimization problem is solved in order to update the design and to predict the non-linear response for the new design. This prediction is then used to accelerate the analysis of the design in the next incremental problem. Each analysis problem is solved by a Newton-type method, and each design problem by a first-order approximation.

The simultaneous solution of analysis and design has also been studied for applications where the analysis problem is stated without any finite element discretization. Hrymak et al. (1985) considered the optimization of extended heat transfer surfaces, and Biegler (1988) used simultaneous approaches to process optimization. Both Hrymak et al. and Biegler applied SQP methods.

In this paper, we develop two algorithms based on the augmented Lagrangian concept for the integrated analysis and design problem. This concept has been successfully applied earlier to many classes of non-linear optimization problems. It is closely related to penalty function techniques as well as ordinary Lagrangian schemes, but avoids the difficulties of both; it removes in particular the ill-conditioning which appears in penalty function methods. We have chosen to use a projected Newton method and a simplicial decomposition scheme, respectively, for solving the subproblems which arise in the augmented Lagrangian schemes. These methods fully exploit the underlying structures of the subproblems.

The contents of the paper are as follows. First, in Section 2 we state an integrated formulation of the problem under consideration, namely linear elastic bar-truss structures. In Section 3, we give a general introduction to the augmented Lagrangian reformulation technique. Then, in Section 4 we derive two applications of the augmented Lagrangian approach to the integrated formulation. In particular, the solution procedures for the subproblems are given in detail. In Section 5, the computational behaviour is analysed. Related methods are discussed in Section 6, and finally, conclusions are drawn and future areas of research are suggested.

2 Problem formulation

In the problem considered, the objective is to minimize the structural weight and the design variables are transverse ar-

\[ f(x) = \sum_{j=1}^{n_2} c_j x_j = c^T x, \]  

where \( n_1 \) is the number of elements in the structure, and \( c_j, j = 1, \ldots, n_1 \), are given constants. The values of the design variables are restricted within given bounds, i.e. to belong to a set

\[ X = \{ x | x \geq x_j \geq \bar{x}_j, j = 1, \ldots, n_1 \}. \]  

Moreover, the structural stiffness matrix can be stated as a linear function of the design variables, i.e.

\[ K(x) = K_0 + \sum_{j=1}^{n_1} x_j K_j, \]  

where \( K_j, j = 0, \ldots, n_1 \), are constant positive semi-definite symmetric matrices. For each \( x \in X \), the structure is assumed to be non-degenerate so that the resulting matrix \( K(x) \) is positive definite and symmetric. Additional constraints are limits on stresses and nodal displacements, \( u \); the latter, which are the analysis variables, are restricted by given bounds, i.e. to belong to a set

\[ U = \{ u | u_j \leq u_j \leq \bar{u}_j, j = 1, \ldots, n_2 \}, \]  

where \( n_2 \) is the number of nodal displacement variables. Given the nodal displacements, the stresses in the elements can be calculated as \( \sigma = Ee = E(\nabla N)u \), where \( N \) are the so-called form-functions; furthermore, for trusses, \( E \) and \( \nabla N \) are constant matrices (see e.g. Cook 1981). The stress constraints may, therefore, be expressed as a set of linear inequalities \( Qu \leq b \), where \( Q = (q_1, \ldots, q_m)^T \), \( b \) are given limits on stresses, and \( m \) is the number of stress constraints. The external load acting upon the structure is given by the vector \( p \). The integrated analysis and design problem can now be stated as the non-linear optimization problem

\[ \min_{x, u} f(x) = c^T x, \quad s.t. \left\{ \begin{array}{l} Qu \leq b \\ K(x)u = p \\ u \in U \\ x \in X. \end{array} \right. \]  

Due to the state equations, this formulation is, in general, a non-convex optimization problem, which may have multiple local optima. An important property of the integrated formulation is that all the derivatives needed in solution schemes may be explicitly and analytically calculated. It is also worth noting that displacement constraints handled implicitly in the nested approach are now stated explicitly as lower and upper bounds on the analysis variables.

3 An augmented Lagrangian scheme

The augmented Lagrangian technique is known to be an efficient tool for general non-linear problems; it generates a sequence of simply constrained subproblems, and ill-conditioning is avoided in comparison with the penalty methods to which it is closely related. In this section we give a brief general description of an augmented Lagrangian scheme.

Consider the general non-linear programming problem

\[ \min_{y} f(y), \quad s.t. h(y) = 0, \]