Optimization criteria and algorithms for bar structures made of work-hardening elasto-plastic materials

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Abstract The problems of the optimization of bar structures made of work-hardening elasto-plastic materials are investigated on the basis of the deformation theory of plasticity. Two optimization criteria are discussed - the minimum of the total complementary energy and the minimum of the total elasto-plastic strain potential energy. It is proved that these criteria are equivalent and lead to equally-stressed structures, if the structures are statically determinate. Furthermore, a problem of the minimization of the integral over the structural volume of an arbitrary monotonously increasing strictly convex smooth function of the absolute value of a strain or a stress with prescribed volume of material for statically determinate bar structures also proves to lead to equally-stressed structures. On the basis of the results obtained, two algorithms for the optimization of statically indeterminate trusses are proposed and illustrated. It is noted that the example structure degenerates to a statically determinate one as a result of the optimization process.

1 Introduction
Optimization problems for bar structures made of elasto-plastic materials have been investigated considerably during the last forty years. The most developed approach is the one with perfect (elasto) plastic behaviour of the material. This approach was initiated by Drucker and Shield (1956) who proposed the energy optimization criterion (Drucker-Shield criterion). Recently, optimization problems for structures made of work-hardening elasto-plastic materials have begun to develop rapidly. Both geometric and physical nonlinearities are being investigated intensively. Papers by Cardoso and Arora (1988), Choi and Santos (1987), Tsay and Arora (1989, 1990a,b) and others are devoted to various structural optimization problems where strains and displacements are large and the stress-strain law is nonlinear. Problems with large displacements and small strains are considered by Mróz et al. (1985). Papers by Bendse and Sokołowski (1987), Cinquini and Contro (1987), Dems and Mróz (1978), Kaneko and Maier (1981), Mróz et al. (1985), Nakamura and Takekawa (1989), and Selyugin (1991) deal with physical nonlinearity only. In most of the above-mentioned papers the optimization algorithms are developed on the basis of the sensitivity analysis of optimization problems. These algorithms are general and, therefore, rather complicated. The use of such algorithms requires a great deal of computer resources.

At the same time, Dems and Mróz (1978), Nakamura and Takekawa (1989) and Selyugin (1991) use assumptions of the deformation theory of plasticity (that is, displacements and strains are small) and replace the complicated constraint system and objective function by some energy optimization criteria based on the concept of structural compliance. Actually this formulation of the problem takes into account physical nonlinearity only. It is well-known that energy optimization criteria prove to be advantageous for the development of efficient numerical algorithms in linear-elastic problems (Gellatly and Berke 1974). Therefore, it is natural to hope that this approach will also be fruitful in nonlinear problems.

Dems and Mróz (1978) proposed the total complementary energy minimum criterion with prescribed volume of material as an optimization criterion, and Selyugin (1991) proposed the total elasto-plastic strain potential energy minimum criterion with prescribed volume of material. It was supposed that these criteria led to a minimum of structural compliance.

The total structure complementary energy minimization problem has the form (Dems and Mróz 1978)
\[ \overline{R} = \int_{V} R \, dV \rightarrow \min, \quad \int_{V} dV = V_0 = \text{const.}, \tag{1} \]
where \( R \) is the complementary energy (Kachanov 1974),
\[ d\overline{R} = \varepsilon_{ij} \, d\sigma_{ij}, \]
where \( \overline{R} \) is the total complementary energy, \( V_0 \) is the prescribed volume of the material, \( d \) is the differential symbol, \( \sigma_{ij} \) is the stress tensor, \( \varepsilon_{ij} \) is the strain tensor and \( V \) is the volume, filled by the material.

In accordance with Selyugin (1991) the total elasto-plastic strain potential energy minimization problem is
\[ \overline{H} = \int_{V} H \, dV \rightarrow \min, \quad \int_{V} dV = V_0, \tag{2} \]
where \( \overline{H} \) is the elasto-plastic strain potential energy (Kachanov 1974)
\[ d\overline{H} = \sigma_{ij} \, d\varepsilon_{ij}, \]
\( \overline{H} \) is total elasto-plastic strain potential energy. Here the summation convention concerning repeated letter subscripts is adopted, the coordinate system is Cartesian. The varying parameters in (1) and (2) are structural parameters. Problems (1) and (2) for bar structures will be investigated in this paper in order to develop efficient optimization algorithms.

2 Optimization of statically determinate bar structures
Let us consider (1) and (2) for statically determinate bar structures (trusses, beams) made of work-hardening elasto-plastic materials.
It is supposed that the cross-section in the beams is idealized (Chyras 1989), that is the beams are I-beams with caps of equal areas $F/2$ and thin web of height $h$, which does not bend. The loads and the heights are distributed smoothly along the beams, longitudinal forces are absent. It is assumed that the stress-strain relation $\sigma - \epsilon$ is the same for both tension and compression of the caps.

It is supposed that the trusses are loaded on nodes only and that the bars work in tension. The latter assumption is valid, for example, for cable structures. It is assumed that all the members of the structures are made of an isotropic material. The cross-sectional areas of the bars are varied in the problems considered (in the case of beams these areas are considered to be smooth functions of a longitudinal coordinate along the beams). For (1) and (2), respectively, the Euler equations are

$$\frac{d(R_i F_i)}{dF_i} + \lambda = 0, \quad i = 1, \ldots, n,$$

(3)

$$\frac{d(H_i F_i)}{dF_i} + \mu = 0, \quad i = 1, \ldots, n,$$

(4)

where $i$ is the number of the structural member, $n$ is the total number of the members, $F_i, R_i, H_i$ are the cross-sectional area, the complementary energy and the elasto-plastic potential strain energy for the $i$-th member, respectively, $\lambda, \mu$ are Lagrangian multipliers. Hereinafter the subscript $i$ will be used to denote that some parameters correspond to the $i$-th member of the $n$-member structure, where $n = 1$ for beams. Note that $F, R$ and $H$ in (3) and (4) are functions of a longitudinal coordinate along the beams.

Because the structure is statically determinate, we have for trusses and beams, respectively,

$$\sigma_i = \frac{P_i}{F_i h_i}, \quad i = 1, \ldots, n,$$

(5)

$$\sigma_i = \frac{2M_i}{F_i h_i}, \quad i = 1, \ldots, n,$$

(6)

where $P_i$ is the longitudinal force, $M_i$ is the bending moment, $\sigma_i$ is the stress, $h_i$ is the height, $i = 1, \ldots, n$.

Transforming (3) and (4) and taking into account (5) and (6), we obtain, respectively,

$$-\sigma_i \frac{dR_i}{d\sigma_i} + R_i + \lambda = 0, \quad i = 1, \ldots, n,$$

(7)

$$-\sigma_i \frac{dH_i}{d\sigma_i} + H_i + \mu = 0, \quad i = 1, \ldots, n.$$  

(8)

Taking into account that

$$\frac{dR}{d\sigma} = \epsilon, \quad \sigma \epsilon - R = H,$$

we obtain from (7) the necessary optimality condition in (1) in the form

$$H_i = \lambda, \quad i = 1, \ldots, n,$$

(9)

where $\lambda$ is positive because $H(\sigma)$ is positive.

It follows from (9) that, being equally-stressed, the structure will be optimal in accordance with (1), with the form of the stress-strain relation not being essential. Let us consider (8) in detail. It is supposed that the function $\sigma(\epsilon)$ is smooth. It is known that this function is a concave one (Kachanov 1974). Then $H(\sigma) = H[\sigma(\epsilon)]$ will be a convex function of $\sigma$. In this case the term $\sigma \frac{dH}{d\sigma}$ is a Legendre transformation (Ioffe and Tikhomirov 1975) of the function $H(\sigma)$, that is a function $H^*,$ conjugate to $H(\sigma)$. This conjugate function will be a monotonously increasing positive function of $|\sigma|$. Thus (8) may be written in the form

$$H_i = \mu, \quad i = 1, \ldots, n.$$  

(10)

It is obvious that (10) has a unique solution for any $i$, $|\sigma_i| = \text{const.}, \quad i = 1, \ldots, n$.  

(11)

This solution proves that the optimal structure according to (2) is an equally-stressed structure. Let us show that (9) and (11) are sufficient conditions for structural optimality in accordance with (1) and (2), respectively.

Calculating the second variation of the Lagrangian function associated with (1) (see Ioffe and Tikhomirov 1975), we obtain for beams and trusses, respectively,

$$\delta^2 \left[ \bar{R} + \lambda \left( \int_V dV - V_0 \right) \right] = \int_L \left( \frac{d^2R}{d\sigma^2} \right) (\delta \epsilon)^2 dx,$$

$$\delta^2 \left[ \bar{R} + \lambda \left( \int_V dV - V_0 \right) \right] = \sum_{i=1}^n \left( \frac{d^2R_i}{d\sigma_i^2} \right) (\delta \epsilon_i)^2 \ell_i,$$

and, similarly, for (2)

$$\delta^2 \left[ \bar{H} + \mu \left( \int_V dV - V_0 \right) \right] = \int_L \left( \frac{d^2H}{d\sigma^2} \right) (\delta \epsilon)^2 dx,$$

$$\delta^2 \left[ \bar{H} + \mu \left( \int_V dV - V_0 \right) \right] = \sum_{i=1}^n \left( \frac{d^2H_i}{d\sigma_i^2} \right) (\delta \epsilon_i)^2 \ell_i,$$

where the integration along the contour $L$ for beams is understood as an integration along the bar, $\ell_i$ are the bar lengths, $i = 1, \ldots, n$, $\delta$ and $\delta^2$ are the first and second variation symbols, respectively. Obviously, these second variations will be positive when the following conditions are valid for (1):

$$\frac{d^2R_i}{d\sigma_i^2} > 0, \quad i = 1, \ldots, n,$$

(12)

and, similarly, for (2):

$$\frac{d^2H_i}{d\sigma_i^2} > 0, \quad i = 1, \ldots, n,$$

(13)

The function $R(\sigma)$ is (strictly) convex by definition (Kachanov 1974) and its second derivative is always positive. Therefore, (12) will take place if (9) is valid. Thus we have proved that the equal stress level is a necessary and sufficient condition for optimality according to (1) of the statically determinate bar structures made of work-hardening elasto-plastic material.

Obviously, (13) will be valid if we take into account the concavity of $\sigma(\epsilon)$ and, consequently, the convexity of $\Pi(\sigma)$. Therefore, the equal stress level of statically determinate bar structures is a necessary and sufficient condition for optimality according to (2).

To summarize, we may say that problems (1) and (2) are equivalent for statically determinate bar structures made of work-hardening elasto-plastic material, and lead to an equally-stressed structure. Furthermore, it is simple to see from the above proof that any problem, like (2), leads to an