Optimal layout theory: analytical solutions for elastic structures with several deflection constraints and load conditions

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Abstract The aim of this note is to outline briefly, in the context of trusses, an optimal layout theory for elastic structures involving deflection constraints for several load conditions. The proposed theory is verified by a simple example, for which the solution by other methods is already known.

1 Introduction

Some recent research into topology optimization (e.g. Diaz and Bendse 1992) has drawn attention to the lack of explicit analytical solutions for several load conditions in this field. In the past, the theory of optimal structural layouts of grid-like structures (e.g. Rozvany 1989, Chapt. 8) was used mostly for plastic design. For trusses and grillages of given depth under a single load condition, the same optimal layouts were shown to be valid for elastic design with a stress or compliance constraint (i.e. given total external work). In this note, an extension of the above theory to elastic design involving deflection constraints for several load conditions is given.

Similarly to the original layout theory by Prager and the author (e.g. Prager and Rozvany 1977), the proposed approach is based on the concept of a structural universe (or "ground structure", containing all potential members) and continuum-based optimality criteria (COC, see e.g. Rozvany 1989). It is important to emphasize that the only difference between cross-section or sizing optimization and layout optimization by the above method is that in the latter the cross-sectional area over entire members can take on a zero value and hence optimality criteria for vanishing members are also required.

The proposed theory will be introduced in the context of trusses subject to displacement constraints but it can readily be applied to other structures and additional (e.g. stress) constraints.

2 Optimality criteria

Let the member force in member i for the load condition k be $F_{ik}$, the member force for the virtual load associated with the $j_k$-th active displacement condition under the k-th load $\bar{F}_{ijk}$, the corresponding cross-sectional area $A_i$ with the lower limit $A_{io}$, the member length $L_i$, Young's modulus $E_i$ and the specific weight of material $\gamma_i$. Extending the Kuhn-Tucker conditions to several loading conditions but without stress constraints, an earlier derivation (Rozvany and Zhou 1991, pp. 46-47) leads to

$$\epsilon_{ik} = \frac{F_{ik}}{E_i A_i} , \quad \bar{\epsilon}_{ik} = \frac{\sum_j \nu_{jk} \bar{F}_{ijk}}{E_i A_i} ,$$

$$A_i = \sqrt{\frac{\sum_k F_{ik} \sum_j \nu_{jk} \bar{F}_{ijk}}{E_i \gamma_i (1 - \beta_i)}} , \quad (1)$$

with

$$\beta_i = 0 \quad \text{for} \quad A_i > A_{io} , \quad \beta_i \geq 0 \quad \text{for} \quad A_i = A_{io} , \quad (2)$$

where $\epsilon_{ik}$ is the strain in the i-th member of the "real" truss and $\bar{\epsilon}_{ik}$ is the strain in the "adjoint" truss, both for the load condition k. For the case $A_{io} = 0$, (1) and (2) imply

$$(E_i/\gamma_i) (\sum_k \epsilon_{ik} \bar{\epsilon}_{ik} \leq 1 \quad \text{for} \quad A_i = 0) , \quad (3)$$

$$(E_i/\gamma_i) (\sum_k \epsilon_{ik} \bar{\epsilon}_{ik} = 1 \quad \text{for} \quad A_i > 0) . \quad (4)$$

Since $\bar{F}_{ijk}$ in (1) is nondimensional and $\nu_{jk}$ has the dimension force/length, it is easy to show that the LHS’s of (3) and (4) are nondimensional.

The optimality criteria (3) and (4) provide necessary conditions for minimum truss weight, including layout optimization. In setting up a structural universe, values of $E_i$ and $\gamma_i$ must be allocated to all potential members. The strains in (3) and (4) must be kinematically admissible. Moreover, the strains in (4) for non-vanishing members must satisfy the first two equations under (1), in which $F_{ik}$ and $\bar{F}_{ik}$ equilibrate the real and virtual (adjoint) loads, respectively.

3 Illustrative example

We consider two alternate loads at a distance L from a vertical supporting line, both of which are enclosing 45° with the horizontal (Figs. 1a and b). In this problem, the structural universe consists of members in all possible directions at all points of the half-space to the right of the supporting line. For both loading conditions, the deflection is constrained to a value of $\Delta$ in the direction of the loads. The corresponding adjoint (vertical) loads are shown in Figs. 1c and d. The optimal layout for this problem is given in Fig. 1e, in which the optimal angles take on a value $\alpha_{opt} = 35.26438968$, with $\tan \alpha_{opt} = 1/\sqrt{2}$.

First the correct member sizes for the above displacement constraints will be calculated and then it will be shown that the considered solution satisfies the optimality conditions in (3) and (4). By Fig. 1f, the member forces for the vertical and
Fig. 1. Illustrative example

horizontal components \(P/\sqrt{2}\) of the external load in Fig. 1a
have the magnitude \(P/(2\sqrt{2}\sin \alpha)\) and \(P/(2\sqrt{2}\cos \alpha)\), giving
the member forces \(F_{11} = F_{22} = (P/2\sqrt{2})(1/\cos \alpha - 1/\sin \alpha)\)
and \(F_{12} = F_{21} = (P/2\sqrt{2})(1/\cos \alpha + 1/\sin \alpha)\). Moreover,
for the adjoint loads in Figs. 1c and d, the member forces are
\(\bar{F}_{11} = \bar{F}_{22} = (1/2\sqrt{2})(1/\cos \alpha - 1/\sin \alpha)\)
and \(\bar{F}_{12} = \bar{F}_{21} = (1/2\sqrt{2})(1/\cos \alpha + 1/\sin \alpha)\). Due to symmetry of the solution
we shall adopt \(A_1 = A_2 = A\), \(E_1 = E_2 = E\)
and \(\gamma_1 = \gamma_2 = \gamma\). This means that the displacement caused by
the above load becomes
\[
\Delta = \frac{PL}{8EA \cos \alpha} \left( \frac{1}{\cos \alpha} + \frac{1}{\sin \alpha} \right)^2 + \left( \frac{1}{\cos \alpha} - \frac{1}{\sin \alpha} \right)^2
\]
\[
= \frac{PL}{4EA \cos \alpha} \left( \frac{1}{\sin^2 \alpha} + \frac{1}{\cos^2 \alpha} \right) = \frac{PL}{4EA \cos \alpha \sin^2 \alpha \cos^2 \alpha} = \frac{PL}{E \Delta \cos \alpha \sin^2 (2\alpha)},
\]
where \(E\) is the weight of the truss.

The optimal value of \(\alpha\) can be obtained from the usual stationarity condition
\[
\frac{2\gamma PL^2}{E \Delta} \frac{d(1/\Phi)}{d\alpha} = 0 = -2 \cos \alpha \sin \alpha \sin^2 (2\alpha) + 4 \cos^2 \alpha \sin (2\alpha) \cos (2\alpha) = 0 \Rightarrow \tan^2 \alpha = 1/2,
\]
\[
\sin \alpha = 1/\sqrt{3}, \quad \cos \alpha = \sqrt{2/3}, \quad \alpha_{opt} = 35.2643966^\circ,
\]
\[
\Phi_{opt} = (27/8)(\gamma PL^2/E\Delta), \quad A_{opt} = 9\sqrt{1.5} PL/(E\Delta).
\]

The above proof of optimality within a given topology was obtained by Zhou (Zhou and Rozvany 1991, pp. 328-329).

Comparing the values of \(A\) in (1) and (6), we obtain the values for the Lagrange multiplier \(\nu\)
\[
\nu = (27/16) PL^2 \gamma /E\Delta^2.
\]

Next it will be shown that the above solution satisfies the criteria (3) and (4) for an optimal layout. Considering the loading in Fig. 1a on the truss in Fig. 1e, it can be shown easily that the displacements \(u\) and \(v\) in the \(x\) and \(y\) direction, respectively, at point \(A\) become
\[
u_A = KL, \quad v_A = 2KL,
\]
with
\[
K = \frac{3\sqrt{3}}{8EA}.
\]

In order to prove the optimality of the above layout, we adopt the following real and adjoint displacement fields over the entire half-space in Fig. 1:
\[
u = K P x, \quad v = 2K P x, \quad \bar{u} = K \nu, \quad \bar{v} = 2K \nu.
\]

Considering the above real displacement field, the strain along an arbitrary line segment \(AB\) of the half-space can be calculated on the basis of Fig. 2a, in which the point \(B\) has a zero displacement and the point \(A\) moves to the new position \(A'\). Since the strain is constant along the considered line segment, it is given by (Fig. 2a)
\[
\varepsilon_{11} = K P (2 \sin \alpha + \cos \alpha) \cos \alpha.
\]

The corresponding strain for the direction \(AC\) (or for the member \(AB\) under the second load condition) then becomes
\[
\varepsilon_{12} = K P (-2 \sin \alpha + \cos \alpha) \cos \alpha,
\]
and in the corresponding strain expressions for the adjoint truss \((\bar{\varepsilon}_1\) and \(\bar{\varepsilon}_2)\) \(K P\) is replaced by \(K r\).

Substituting (11) and (12) into (3), we have (with \(\nu_1 = \nu_2 = \nu\) due to symmetry)
\[
\frac{E}{\gamma} (\varepsilon_{11} \varepsilon_{11} + \varepsilon_{12} \varepsilon_{12}) = 2K^2 P \nu (E/\gamma)(4 \sin^2 \alpha \cos^2 \alpha + \cos^4 \alpha) = 2K^2 \nu (E/\gamma) [\sin^2 (2\alpha) + \cos^4 \alpha] .
\]

First we shall show that the extremum of the above expression with respect to \(\alpha\) gives the same member directions as in Fig. 1e. Making the first derivative of the expression in the square bracket in (13) zero,
\[
4 \sin (2\alpha) \cos (2\alpha) - 4 \cos^3 \alpha \sin \alpha = 0 \Rightarrow
\]
\[
\Rightarrow \sin (4\alpha) - 2 \cos^3 \alpha \sin \alpha = 0,
\]
which is satisfied exactly by the \(\alpha\)-value in (6). The variation of the bracketed expression in (13) is shown in Fig. 2b.

It can be checked easily by substituting (7) and (9) into (13) that at \(\alpha_{opt}\) the equality condition (4) is satisfied. Moreover, (14) and Fig. 2b show that at other values of \(\alpha\) the inequality in (3) is fulfilled. Since this is essentially a compliance problem for two load conditions, it is convex and hence (3) and (4) represent necessary and sufficient conditions for minimum truss weight. Global optimality of the layout solution in Fig. 1e has, therefore, been established.