A mixed variational formulation for shape optimization of solids with contact conditions

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Abstract This paper is concerned with the development of a mixed variational formulation and computational procedure for the shape optimization problem of linear elastic solids in possible contact with a rigid foundation. The objective is to minimize the maximum value of the von Mises equivalent stress in a body (non-differentiable objective function), subject to a constraint on its volume and bound constraints on the design. For design purposes, the contact boundary is considered fixed.

A finite element model that is appropriate for the mixed formulation is utilized in the discretization of the state and adjoint state equations. An elliptical mesh generator was used to generate the finite element mesh at each new design. The computational model is tested in several example problems.

1 Introduction

This study is concerned with the development of a variational formulation and procedure for the computational solution for the optimal shape design of a two-dimensional, linear elastic body in possible contact with a rigid rough foundation using a mixed variational formulation of the problem.

The structural optimization problem as treated in this work has the following form. The minimization of the maximum von Mises equivalent stress within a bound on the total volume of the body and local design constraints, depending on the example to be solved. The portion of the boundary where contact between the body and the foundation is possible is assumed to be fixed for design purposes.

Here, we should note the previous works by Benedict and Taylor (1981), Sokolowsky and Zolesio (1987) and Haslinger and Neittaanmäki (1988), where the problem of optimizing the contact interface is treated.

The optimization problem proposed is non-differentiable due to the maximum objective function and to the non-differentiability of the state variable with respect to design. If one adds to it the problem of contact with friction, which in itself continues to be one of the most challenging problems in computational mechanics, it is quite obvious that the degree of complexity achieved prevents the use of analytical methods for its solution in a general setting. So it is clear that the only alternative is to use a numerical discretization method for contact problems.

The finite element method has been established to be one of the most important and versatile tools for analysis in problems of mechanics and, especially, in solid and structural mechanics. In its virtual displacement form, numerous and quite successful formulations of the method for contact problems have been developed and presented in the literature (e.g. Kikuchi and Oden 1988).

In the field of shape optimization, Zienkiewicz and Campbell (1973) were among the first to approach this problem using a virtual displacement-based finite element model. Subsequently, this method has been widely applied to problems in shape optimal design (see, e.g. Dems and Mróz 1978; Pedersen and Laursen 1983; Braibant and Fleury 1984; Rasmussen 1990). The virtual displacement finite element method has two main disadvantages. The increase of finite element error that results from mesh distortion during the shape redesign process, and in some situations a lack of sufficient precision in the prediction of stresses and strains at the boundary and internal nodes.

Within FEM applications, very important work has been presented to avoid some of the disadvantages mentioned above. The domain method proposed by Choi and Seong (1986), where sensitivity expressions are defined in terms of domain integrals rather than boundary integrals, thereby avoiding the evaluation of state variables at the boundary, provides for improved accuracy in the numerical evaluation of sensitivities. Also Haber (1987) presented a Eulerian-Lagrangian formulation based on the mutual Washizu functional, where the shape optimal design problem can be formulated in an arbitrary initial domain as a means to overcome the difficulties inherited from shape redesign.

In this work another approach is considered. With the development of powerful automatic mesh generation and optimization techniques, the first of the cited disadvantages in FEM is avoided. To overcome the other difficulty, it is proposed in this work to extend the mixed variational formulation for shape design presented in the paper by Rodrigues (1989), in which independent interpolations are assumed for the stress, strain and displacement fields, to include contact conditions.

For the contact problem with friction, where accurate evaluation of the stress field at the contact boundary is required to accurately predict the stick-slip regions and the contact pressure, the mixed method seems especially applicable since the stress field is obtained as part of the solution rather than from an a posteriori evaluation based on the gradient of the displacement field. Furthermore, if the displacement based finite element method is applied, the strain and stress fields are computed at each Gaussian integration point. However, values of stress and adjoint strain should be obtained at the design boundary for shape design problems, thus an extrapolation method would be required. The mixed
finite element method avoids this problem since the values for displacement, strain and stress are directly obtained at the nodal points. These considerations are brought together in the developments reported here to demonstrate a more effective approach to the overall treatment of the optimal shape design of structural components.

2 Analytical model

2.1 State equation for two-dimensional, linear elastic bodies with contact interfaces

Consider the two-dimensional, linear elastic body \( \Omega \) initially separated from a rough rigid foundation \( \mathcal{R} \), as described in Fig. 1.

Let us denote by \( u \) a specific displacement field of the body \( \Omega \), which corresponds to the equilibrium state for the given body forces \( b \) and boundary tractions \( f \). Under the assumption of small displacements and assuming a Coulomb friction coefficient \( \delta \), \( u \) is the solution of the following system of equations (classical elasticity model):

\[
\begin{align*}
\sigma_{ij}(u)_{,j} + b_i &= 0 \quad \text{in } \Omega, \\
u_i &= 0 \quad \text{on } \Gamma_u, \\
\sigma_{ij}(u)n_j &= f_i \quad \text{on } \Gamma_t, \\
\sigma_{ij}(u)n_i n_j - g &= 0 \quad \text{on } \Gamma_c, \\
|\sigma_i(u)| &< \delta \quad |\sigma_{ij}(u)n_i n_j| \quad \text{on } \Gamma_c, \\
\sigma_i(u) &= \delta \quad |\sigma_{ij}(u)n_i n_j| \Rightarrow \exists \Omega(x) \geq 0, \quad \text{such that} \\
u_i &= -\lambda \sigma_i(u) \quad \text{on } \Gamma_c.
\end{align*}
\]

Here \( s(x) \) represents the prescribed frictional force and \( \Gamma_c \) is the part of the contact boundary where the frictional force is prescribed.

In (11) the following compact notation is used:

\[
\begin{align*}
D(\sigma, e) &= \int_{\Omega} E_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \, d\Omega, \\
C(\tau, e) &= \int_{\Omega} \gamma_{ij} \varepsilon_{ij} \, d\Omega, \\
B(\tau, \sigma) &= \int_{\Omega} \frac{1}{2}(\varepsilon_{ij} + \varepsilon_{ji}) \, d\Omega, <b, v > = \int_{\Omega} b_i v_i \, d\Omega, \\
< f, v > = \int_{\Omega} f_i v_i \, d\Omega.
\end{align*}
\]

Problem (7)-(10) is just a generalization of the Hu-Washizu variational principle for bodies undergoing unilateral contact with a rigid foundation and with given friction force.

From the first variation of the Lagrangian \( L \) with respect to (w.r.t.) the variables \( v, e, \sigma, \mu_u, \mu_c, \mu_f \), the necessary conditions for the saddle point \( (u, e, \sigma, \mu_u, \mu_c, \mu_f) \) are obtained as presented below.

Stationarity with respect to \( v, e \) and \( \tau \) requires

\[
D(\varepsilon, e) - C(\varepsilon, e) - C(\sigma, e) + B(\sigma, v) + B(\tau, u) - < b, v > - \Omega - \Omega - < f, v > = \int_{\Gamma_t} \mu_u v_i \, d\Gamma + \int_{\Gamma_c} \mu_c v_i n_i \, d\Gamma + \int_{\Gamma_u} \mu_f v_i \, d\Gamma.
\]